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Optimality and Duality in Nondifferentiable and Multiobjective Programming under Generalized d-Invexity*

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Abstract. In this paper, we are concerned with the nondifferentiable multiobjective programming problem with inequality constraints. We introduce four new classes of generalized d-type-I functions. By utilizing the new concepts, Antczak type Karush-Kuhn-Tucker sufficient optimality conditions, Mond-Weir type and general Mond-Weir type duality results are obtained for nondifferentiable and multiobjective programming.

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1. Introduction

Convexity plays a vital role in many aspects of mathematical programming including sufficient optimality conditions and duality theorems see for example Mangasarian (1969) and Bazaraa et al. (1991).

To relax convexity assumptions imposed on the functions in theorems on sufficient optimality and duality, various generalized convexity notions have been proposed. Hanson (1981) introduced the class of invex functions. Later, Hanson and Mond (1987) defined two new classes of functions called type-I and type-II functions, and sufficient optimality conditions were established by using

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these concepts. Rueda and Hanson (1988) further extended type-I functions to the classes of pseudo-type-I and quasi-type-I functions and obtained sufficient optimality criteria for a nonlinear programming problem involving these functions. Kaul et al. (1994) considered a multiple objective nonlinear programming problem involving generalized type-I functions and obtained some results on optimality and duality, where the Wolfe and Mond-Weir duals are considered. Univex functions were introduced and studied by Bector et al. (1992). Rueda et al. (1995) obtained optimality and duality results for several mathematical programs by combining the concepts of type-I and univex functions. Mishra (1998) considered a multiple objective nonlinear programming problem and obtained optimality, duality and saddle point results of a vector valued Lagrangian by combining the concepts of type-I, pseudo-type-I, quasi-type-I, quasi-pseudo-type-I, pseudo-quasi-type-I and univex functions. Aghezzaf and Hachimi (2000) introduced new classes of generalized type-I vector-valued functions and derived various duality results for a nonlinear multiobjective programming problem.

It is known that, despite substituting invexity for convexity, many theoretical problems in differentiable programming can also be solved, see Hanson (1981), Egudo and Hanson (1987), and Jeyakumar and Mond (1992). But the corresponding conclusions cannot be obtained in nondifferentiable programming with the aid of invexity introduced by Hanson (1981) because the existence of a derivative is required in the definition of invexity.

There exists a generalization of invexity to locally Lipschitz functions, with derivative replaced by the Clarke generalized gradient, see Craven (1986), Reiland (1990), Mishra and Mukherjee (1994), Mishra (1996), Mishra (1997), and Mishra and Giorgi (2000). However, Antczak (2002) used directional derivative, in association with a hypothesis of an invex kind following Ye (1991). The necessary optimality conditions in Antczak (2002) are different from those cited in the literature.

In the present paper, we consider a nondifferentiable and multiobjective programming problem and derive some Karush-Kuhn-Tucker type of sufficient optimality conditions for a (weakly) Pareto efficient solution to the problem involving the new classes of directionally differentiable generalized type-I functions. Furthermore, the Mond-Weir type and general Mond-Weir type of duality results are also obtained in terms of right differentials of the aforesaid functions involved in the multiobjective programming problem.

2. Preliminaries

In this section, we extend the concepts of weak strictly-pseudoquasi-type I, strong pseudoquasi-type I, weak quasistrictly-pseudo type I and weak strictly pseudo-type I functions introduced in Aghezzaf and Hachimi (2000) in the setting of Antczak (2002) and give some preliminaries.

Consider the following multiobjective programming problem:

$$\begin{aligned} &\min\ f(x) \\ \text{s.t.}\ \ & g(x)\leqslant 0, \\ & x\in X, \end{aligned} \tag{P}$$

where f: $X\to R^k$, g: $X\to R^m$, X is a nonempty open subset of R^n. Suppose that η: $X\times X\to R^n$ is a vector function. Through this paper, $f'(u,\eta(x,u))=\lim_{\lambda\to 0'}\frac{f(u+\lambda\eta(x,u))-f(u)}{\lambda}$. A similar notation is made for $g'(u,\eta(x,u))$.

Let $D=\{x\in X\colon\ g(x)\leqslant 0\}$ be the set of all the feasible solutions for (P) and denote $I=\{1,\dots,k\}$, $M=\{1,2,\dots,m\}$, $J(x)=\{j\in M: g_j(x)=0\}$ and $\widetilde{J}(x)=\{j\in M: g_j(x)<0\}$. It is obvious that $J(x)\cup\tilde{J}(x)=M$.

Throughout this paper, the following convention for vectors in R^n will be followed:

$$x>y \quad \text{if and only if } x_i>y_i,\quad i=1,2,\dots,n,$$

$$x\geqq y \quad \text{if and only if } x_i\geqslant y_i,\quad i=1,2,\dots,n,$$

$$x\geqslant y \quad \text{if and only if } x_i\geqslant y_i,\quad i=1,2,\dots,n,\quad \text{but } x\neq y.$$

DEFINITION 2.1. (f,g) is said to be d-type-I with respect to η at $u\in X$ if there exists a vector function η such that for all $x\in X$,

$$f(x)-f(u)\geqq f'(u,\eta(x,u))$$

and

$$-g(u)\geqq g'(u,\eta(x,u)).$$

DEFINITION 2.2. (f,g) is said to be weak strictly-pseudo-quasi d-type-I with respect to η at u∈X if there exists a vector function η such that for all $x\in X$,

$$f(x)\leqslant f(u)\Rightarrow f'(u,\eta(x,u))<0$$

and

$$-g(u)\leqq 0\Rightarrow g'(u,\eta(x,u))\leqq 0.$$

DEFINITION 2.3. (f,g) is said to be strong pseudo-quasi d-type I with respect to η at $u\in X$ if there exists a vector function η such that for all $x\in X$,

$$f(x)\leqslant f(u)\Rightarrow f'(u,\eta(x,u))\leqslant 0$$

and

$$-g(u)\leqq 0\Rightarrow g'(u,\eta(x,u))\leqq 0.$$

EXAMPLE 2.1. Consider the function $f=(f_1, f_2)$: $[-1,4)\to R$ defined by

$$f_1=\begin{cases} x^3 & -1\leqslant x<2 \\ 8 & 2\leqslant x<4 \end{cases}$$

$$f_2=\begin{cases} 0 & -1\leqslant x<2 \\ 2x^2-8 & 2\leqslant x<4 \end{cases}$$

and the function $g=(g_1, g_2)$: $[-1,4)\to R$ defined by

$$g_1=\begin{cases} -x^2 & -1\leqslant x<2 \\ -4 & 2\leqslant x<4 \end{cases}$$

$$g_2=\begin{cases} 5x & -1\leqslant x<2 \\ x^4-6 & 2\leqslant x<4. \end{cases}$$

Clearly, f_1, f_2, g_1 and g_2 are not differentiable functions at $x=2$, but only directionally differentiable fucntions at $x=2$. The feasible region is nonempty. Let $\eta(x,\bar{x})=x^2(x-\bar{x})/2$ and $\bar{x}=2$.

(i) If $x\in[-1,2)$ and $f_1(x)+f_2(x)\leqslant f_1(2)+f_2(2)$, then it implies that $x\leqslant 2$, which further implies that $f_1'(\bar{x};\eta)+f_2'(\bar{x};\eta)=6x^2(x-2)\leqslant 0$, and $-g_1(\bar{x})-g_2(\bar{x})\leqslant 0$ implies that $g_1'(\bar{x};\eta)+g_2'(\bar{x};\eta)\leqslant 0$.
(ii) The case $x\in[2,4)$ can be verified similiarly.

Thus (f, g) is strong pseudo-quasi d-type I with respect to η at $x=2$. However, f and g are not d-invex functions at $x=2$ with respect to the same $\eta(x,\bar{x})=x^2(x-\bar{x})/2$.

DEFINITION 2.4. (f, g) is said to be weak quasi-strictly-pseudo d-type-I with respect to η at u$\in$X if there exists a vector funtion η such that for all $x\in X$,

$$f(x)\leqslant f(u)\Rightarrow f'(u,\eta(x,u))\leqq 0$$

and

$$-g(u)\leqq 0\Rightarrow g'(u,\eta(x,u))\leqslant 0.$$

DEFINITION 2.5. (f, g) is said to be weak strictly-pseudo d-type-I with respect to η at $u\in X$ if there exists a vector function η such that for all $x\in X$,

$$f(x)\leqslant f(u)\Rightarrow f'(u,\eta(x,u))<0$$

and

$$-g(u)\leqq 0\Rightarrow g'(u,\eta(x,u))<0.$$

Remark 2.1. The functions defined above are different from those in Suneja et al. (1997), Aghezzaf and Hachimi (2000) and Antczak (2002). For examples of differentiable generalized type functions, one can refer to Aghezzaf and Hachimi (2000).

DEFINITION 2.6. A point $\bar{x} \in D$ is said to be a weak Pareto efficient solution for (P) if the relation

$$f(x) \nless f(\bar{x})$$

holds for all $x \in D$.

DEFINITION 2.7. A point $\bar{x} \in D$ is said to be a locally weak Pareto efficient solution for (P) if there is a neighborhood $N(\bar{x})$ around $\bar{x}$ such that

$$f(x) \nless f(\bar{x})$$

holds for all $x \in N(\bar{x}) \cap D$.

DEFINITION 2.8. A function f: $X \to R$ is said to be preinvex with respect to η on X if

$$f(u + \lambda\eta(x, u)) \leqq \lambda f(x) + (1 - \lambda) f(u)$$

holds for all $x, u \in X$ and $\lambda \in [0, 1]$.

The following results from Antczak (2002) and Weir and Mond (1988) will be needed in the sequel of the paper.

LEMMA 2.1. *If $\bar{x}$ is a locally weak Pareto or a weak Pareto efficient solution of* (P) *and if g_j is continuous at $\bar{x}$ for $j \in \tilde{J}(\bar{x})$, then the following system of inequalities*

$$f'(\bar{x}, \eta(x, \bar{x})) < 0,$$
$$g'_{J(\bar{x})}(\bar{x}, \eta(x, \bar{x})) < 0,$$

has no solution for $x \in X$.

LEMMA 2.2. *Let S be a nonempty set in R^n and ψ: $S \to R^p$ be a preinvex function on S. Then either $\psi(x) < 0$ has a solution $x \in S$, or $\lambda^T \psi(x) \geqslant 0$ for all $x \in S$, or some $\lambda \in R^m_+$, but both alternatives are never true.*

LEMMA 2.3 (Fritz John type necessary optimality condition). *Let $\bar{x}$ be a weak Pareto efficient solution for* (P). *Moreover, we assume that g_j is continuous for $j \in \tilde{J}(\bar{x})$, f and g are directionally differentiable at $\bar{x}$ with $f'(\bar{x}, \eta(x,\bar{x}))$, and $g'_{J(\bar{x})}(\bar{x}, \eta(x,\bar{x}))$ pre-invex functions of x on X. Then there exist $\bar{\xi} \in R^k_+$, $\bar{\mu} \in R^m_+$, such that $(\bar{x}, \bar{\xi}, \bar{\mu})$ satisfies the following conditions*:

$$\bar{\xi}^T f'(\bar{x}, \eta(x,\bar{x})) + \bar{\mu}^T g'(\bar{x}, \eta(x,\bar{x})) \geqslant 0 \quad \forall x \in X,$$

$$\bar{\mu}^T g(\bar{x}) = 0,$$

$$g(\bar{x}) \leqq 0.$$

DEFINITION 2.9. Function g is said to satisfy the generalized Slater's constraint qualification at $\bar{x} \in D$ if g is d-invex at $\bar{x}$, and there exists $\tilde{x} \in D$ such that $g_j(\tilde{x}) < 0$, $j \in J(\bar{x})$.

DEFINITION 2.10. Let $f\colon X \to R^k$ be defined on X and directionally differentiable at $u \in X$. f is said to be d-invex at $u \in X$ with respect to η if for any $x \in X$,

$$f(x) - f(u) \geqq f'(u, \eta(x,u)).$$

LEMMA 2.4 (Karush-Kuhn-Tucker type necessary optimality condition). *Let $\bar{x}$ be a weak Pareto efficient solution for* (P). *Assume that g_j is continuous for $j \in \tilde{J}(\bar{x})$, f and g are directionaly differentiable at $\bar{x}$ with $f'(\bar{x}, \eta(x,\bar{x}))$, and $g'_{J(\bar{x})}(\bar{x}, \eta(x,\bar{x}))$ pre-invex functions of x on X. Moreover, we assume that g satisfies the generalized Slater's constraint qualification at $\bar{x}$. Then there exists $\bar{\mu} \in R^m_+$ such that $(\bar{x}, \bar{\mu})$ satisfies the following conditions*:

$$f'(\bar{x}, \eta(x,\bar{x})) + \bar{\mu}^T g'(\bar{x}, \eta(x,\bar{x})) \geqq 0, \quad \forall x \in X, \tag{1}$$

$$\bar{\mu}^T g(\bar{x}) = 0, \tag{2}$$

$$g(\bar{x}) \leqq 0. \tag{3}$$

3. Sufficient Optimality Conditions

In this section, we establish a Karush-Kuhn-Tucker type sufficient optimality condition.

THEOREM 3.1. *Let $\bar{x}$ be a feasible solution for* (P) *at which conditions* (1)–(3) *are satisfied. Moreover, if any of the following conditions is satisfied*:

(a) $(f, \mu^T g)$ *is strong pseudoquasi d-type-I at $\bar{x}$ with respect to η*;
(b) $(f, \mu^T g)$ *is weak strictly pseudoquasi d-type-I at $\bar{x}$ with respect to η*;
(c) $(f, \mu^T g)$ *is weak strictly pseudo d-type-I at $\bar{x}$ with respect to η,*

then $\bar{x}$ is a weak Pareto efficient solution for (P).

Proof. We proceed by contradiction. Suppose that $\bar{x}$ is not a weak Pareto efficient solution of (P). Then there is a feasible solution x of (P) such that

$$f_i(x) < f_i(\bar{x}), \quad \text{for any } i \in \{1, 2, \ldots, k\}. \tag{4}$$

By condition (a) and (2), we get

$$f'(\bar{x}, \eta(x, \bar{x})) < 0$$

and

$$\bar{\mu}^T g'(\bar{x}, \eta(x, \bar{x})) \leqslant 0.$$

By these two inequalities, we get

$$f'(\bar{x}, \eta(x, \bar{x})) + \bar{\mu}^T g'(\bar{x}, \eta(x, \bar{x})) < 0,$$

which contradicts (1).
By condition (b), from (4) and (2), we get

$$f'(\bar{x}, \eta(x, \bar{x})) + \bar{\mu}^T g'(\bar{x}, \eta(x, \bar{x})) < 0,$$

again a contradiction to (1).
By condition (c), from (4) and (2), we get

$$f'(\bar{x}, \eta(x, \bar{x})) < 0$$

and

$$\bar{\mu}^T g'(\bar{x}, \eta(x, \bar{x})) < 0.$$

By these two inequalities, we get

$$f'(\bar{x}, \eta(x, \bar{x})) + \bar{\mu}^T g'(\bar{x}, \eta(x, \bar{x})) < 0,$$

which contradicts (1). This completes the proof. □

EXAMPLE 3.1. Consider function $f = (f_1, f_2)$ defined on $X = R$, by $f_1(x) = x^2$ and $f_2(x) = x^3$ and function g defined on $X = R$, by

$$g = \begin{cases} -2x^2, & -1 \leqslant x < 2 \\ -x^3, & 2 \leqslant x < 2.5. \end{cases}$$

Clearly, g is not differentiable at $x = 2$, but only directionally differentiable at $x = 2$. The feasible region is nonempty. Let $\eta(x, \bar{x}) = (x - \bar{x})/2$ and $\bar{x} = 0$.

(i) If $x \in [-1,2), -g_1(\bar{x}) = 0$, implies that $g'(\bar{x};\eta) = 0$.
(ii) The case $x \in [2,4)$ can be verified similiarly.

$$f(x) \leqq f(\bar{x}) \Rightarrow \nabla f(\bar{x})\eta(x,\bar{x}) = 0, \quad \text{for all } x.$$

Thus (f, g) is strong pseudo-quasi d-type I at $x = 0$. But, f and g are not d-invex functions at $x = 0$ with respect to the same $\eta(x,\bar{x}) = (x - \bar{x})/2$. Therefore, Theorem 13 of Antczak (2002) is not applicable. Then, by Theorem 3.1(a), $\bar{x}$ is a weak Pareto solution for the given multiobjective programming problem.

4. Mond-Weir Duality

Now, in relation to (P) we consider the following dual problem, which is in the format of Mond-Weir (1981):

$$\begin{aligned} \text{(MWD)} \quad \max \quad & f(y) = (f_1(y), f_2(y), \ldots, f_k(y)) \\ \text{s.t.} \quad & (\xi^T f' + \mu^T g')(y, \eta(x,y)) \geqq 0, \quad \text{for all } x \in D, \qquad (5) \\ & \mu_j g_j(y) \geqslant 0, \quad j = 1, \ldots, m, \end{aligned}$$

$$\xi^T e = 1, \tag{6}$$

$$\xi \in R^k_+, \quad \mu \in R^m_+, \tag{7}$$

where $e = (1, 1, \ldots, 1) \in R^k$.

Let

$$W = \left\{ \begin{array}{l} (y, \xi, \mu) \in X \times R^k \times R^m : (\xi^T f' + \mu^T g')(y, \eta(x,y)) \geqq 0, \\ \mu_j g_j(y) \geqslant 0, j = 1, \ldots, m, \xi \in R^k_+, \xi^T e = 1, \mu \in R^m_+ \end{array} \right\}$$

denote the set of all the feasible solutions of (MWD).

We denote by $\mathrm{pr}_X W$ the projection of set W on X.

THEOREM 4.1 (Weak duality). *Let x and (y, ξ, μ) be feasible solutions for* (P) *and* (MWD), *respectively. Moreover, we assume that any one of the following conditions holds*:

(a) $(f, \mu^T g)$ *is strong pseudoquasi d-type-I at y with respect to η and $\xi > 0$*;
(b) $(f, \mu^T g)$ *is weak strictly pseudoquasi d-type-I at y with respect to η*;
(c) $(f, \mu^T g)$ *is weak strictly pseudo d-type-I at y with respect to η at y on $D \cup \mathrm{pr}_X W$*.

Then the following cannot hold:

$$f(x) \leqslant f(y).$$

Proof. We proceed by contradiction. Suppose that

$$f(x) \leqslant f(y). \tag{8}$$

Since (y, ξ, μ) is feasible for (MWD), if follows that

$$-\sum_{j=1}^{m} \mu_j g_j(y) \leqslant 0. \tag{9}$$

By condition (a), (8) and (9) imply

$$f'(y, \eta(x, y)) \leqslant 0, \tag{10}$$

$$\sum_{j=1}^{m} \mu_j g_j'(y, \eta(x, y)) \leqslant 0. \tag{11}$$

Since $\xi > 0$, the above two inequalities give

$$\sum_{i=1}^{k} \xi_i f_i'(y, \eta(x, y)) + \sum_{i=1}^{m} \mu_j g_j'(y, \eta(x, y)) < 0, \tag{12}$$

which contradicts (5).
By condition (b), (8) and (9) imply

$$f'(y, \eta(x, y)) < 0, \tag{13}$$

$$-\sum_{j=1}^{m} \mu_j g_j(y) \leqslant 0. \tag{14}$$

Since $\xi \geqslant 0$, (13) and (14) imply (12), again a contradiction to (5).
By condition (c), (8) and (9) imply

$$f'(y, \eta(x, y)) < 0, \tag{15}$$

$$-\sum_{j=1}^{m} \mu_j g_j(y) < 0. \tag{16}$$

Since $\xi \geqslant 0$, (15) and (16) imply (12), again a contradiction to (5). This completes the proof. □

THEOREM 4.2 (Strong duality). *Let $\bar{x}$ be a locally weak Pareto efficient solution or weak Pareto efficient solution for* (P) *at which the generalized Slater's constraint qualification is satisfied, f, g be directionally differentiable at $\bar{x}$ with $f'(\bar{x}, \eta(x, \bar{x}))$, and $g'(\bar{x}, \eta(x, \bar{x}))$ preinvex functions on X and g_j be continuous for $j \in \hat{J}(\bar{x})$. Then there exists $\bar{\mu} \in R_+^m$ such that $(\bar{x}, 1, \bar{\mu})$ is feasible for* (MWD). *If the weak duality between* (P) *and* (MWD) *in Theorem* 4.1 *holds, then $(\bar{x}, 1, \bar{\mu})$ is a locally weak Pareto efficient solution for* (MWD).

Proof. Since $\bar{x}$ satisfies all the conditions of Lemma 2.4, there exists $\bar{\mu} \in R_+^m$ such that conditions (1)–(3) hold. By (1)–(3), we have that $(\bar{x}, 1, \bar{\mu})$ is feasible for (MWD). Also, by the weak duality, if follows that $(\bar{x}, 1, \bar{\mu})$ is a locally weak Pareto efficient solution for (MWD). □

THEOREM 4.3 (Converse duality). *Let* $(\bar{y}, \bar{\xi}, \bar{\mu})$ *be a weak Pareto efficient solution for* (MWD). *Moreover, we assme that the hypothesis of Theorem* 3.1 *hold* $\bar{y}$ *in* $D \cup \mathrm{pr}_X W$. *Then* $\bar{y}$ *is a weak Pareto efficient solution for* (P).

Proof. We proceed by contradiction. Suppose that $\bar{y}$ is not a weak Pareto efficient solution for (P), that is, there exists $\tilde{x} \in D$ such that $f(\bar{x}) < f(\bar{y})$. Since condition (a) of Theorem 4.1 holds, we get

$$\sum_{i=1}^{k} \bar{\xi}_i f_i'(\bar{y}, \eta(\tilde{x}, \bar{y})) < 0. \tag{17}$$

From the feasibility of $\tilde{x}$ for (P) and $(\bar{y}, \bar{\xi}, \bar{\mu})$ for (MWD) respectively, we have

$$-\sum_{j=1}^{m} \bar{\mu}_j g_j(\bar{y}) \leqslant 0,$$

which in light of condition (a) of Theorem 4.1 yields

$$\sum_{j=1}^{m} \bar{\mu}_j g_j'(\bar{y}, \eta(\tilde{x}, \bar{y})) \leqslant 0. \tag{18}$$

By (17) and (18), we get

$$\sum_{i=1}^{k} \bar{\xi}_i f_i'(\bar{y}, \eta(\tilde{x}, \bar{y})) + \sum_{j=1}^{m} \bar{\mu}_j g_j'(\bar{y}, \eta(\tilde{x}, \bar{y})) < 0. \tag{19}$$

This contradicts the dual constraint (5).

By condition (b), we get

$$\sum_{i=1}^{k} f_i'(\bar{y}, \eta(\tilde{x}, \bar{y})) < 0$$

and

$$\sum_{j=1}^{m} \bar{\mu}_j g_j'(\bar{y}, \eta(\tilde{x}, \bar{y})) \leqslant 0.$$

Since $\bar{\xi}_i \geqslant 0$, the above two inequalities imply (19), again a contradiction to (5). By condition (c), we have

$$\sum_{i=1}^{k} f_i'(\bar{y}, \eta(\tilde{x}, \bar{y})) < 0$$

and

$$\sum_{j=1}^{m} \bar{\mu}_j g_j'(\bar{y}, \eta(\tilde{x}, \bar{y})) < 0.$$

Since $\bar{\xi} \geqslant 0$, the above two inequalities imply (19), again a contradiction to (5). This completes the proof. □

5. General Mond-Weir Duality

We shall continue our discussion on duality for (P) in the present section by considering a general Mond-Weir type dual problem of (P) and proving weak and strong duality theorems under an assumption of the generalized d-invexity introduced in Section 2.

We consider the following general Mond-Weir type dual to (P)

$$\begin{aligned} \text{(GMWD)} \quad \max \quad & \phi(y,\xi,\mu)=f(y)+\mu_{J_0}^T g_{J_0}(y)e \\ \text{s.t.} \quad & (\xi^T f'+\mu^T g')(y,\eta(x,y))\geqq 0, \quad \text{for all } x\in D, && (20)\\ & \mu_{J_1} g_{J_1}(y)\geqslant 0, 1\leqslant t\leqslant r && (21)\\ & \xi^T e=1, && (22)\\ & \xi\in R_+^k, \quad \mu\in R_+^m, \end{aligned}$$

where $J_t,\ 0\leqslant t\leqslant r$ are partitions of set M and $e=(1,1,\dots,1)\in R^k$. Let

$$\widetilde{W}=\left\{ \begin{array}{l} (y,\xi,\mu)\in X\times R^k\times R^m\colon (\xi^T f'+\mu^T g')(y,\eta(x,y))\geqq 0, \\ \mu_j g_j(y)\geqslant 0, j=1,\dots,m, \xi\in R_+^k, \xi^T e=1, \mu\in R_+^m \end{array} \right\}$$

denote the set of all the feasible solutions of (MWD).

THEOREM 5.1 (Weak duality). *Let x and (y,ξ,μ) be feasible solutions for* (P) *and* (GMWD) *respectively. If any one of the following conditions holds*:

(a) $\xi>0$, *and* $(f+\mu_{J_0}g_{J_0},\mu_{J_t}g_{J_t})$ *is strong pseudo d-type-I at y in* $D\cup \text{pr}_X\tilde{W}$, *with respect to η for any* $t, 1\leqslant t\leqslant r$;

(b) $(f+\mu_{J_0}g_{J_0},\mu_{J_t}g_{J_t})$ *is weak strictly pseudoquasi d-type-I at y in* $D\cup \text{pr}_X\tilde{W}$, *with respect to η for any* $t,\ 1\leqslant t\leqslant r$;

(c) $(f+\mu_{J_0}g_{J_0},\mu_{J_t}g_{J_t})$ *is weak strictly pseudo d-type-I at y in* $D\cup \text{pr}_X\tilde{W}$, *with respect to η for any* $t,\ 1\leqslant t\leqslant r$,

then the following cannot hold:

$$f(x)\leqslant \phi(y,\xi,\mu).$$

Proof. We proceed by contradiction. Suppose that

$$f(x)\leqslant \phi(y,\xi,\mu). \tag{23}$$

Since x is feasible for (P) and $\mu\geqslant 0$, (23) implies that

$$f(x)+\mu_{J_0}^T g_{J_0}(x)e\leqslant f(y)+\mu_{J_0}^T g_{J_0}(y)e. \tag{24}$$

From (21), we have

$$-\mu_{J_t}^T g_{J_t}(y)\leqq 0, \quad \text{for all } 1\leqslant t\leqslant r. \tag{25}$$

By condition (a), from (24) and (25), we have

$$(f' + \mu_{J_0} g'_{J_0} e)(y, \eta(x, y)) \leqslant 0,$$

and

$$\mu_{J_t} g'_{J_t}(y, \eta(x, y)) \leqslant 0, \quad \forall 1 \leqslant t \leqslant r.$$

Since $\xi > 0$, the above two inequalities yield

$$\left(\xi^T f' + \sum_{t=0}^{r} \mu_{J_t} g'_{J_t} \right)(y, \eta(x, y)) < 0. \tag{26}$$

Since $J_0, \ldots, J_r$ are partitions of M, (26) is equivalent to

$$(\xi^T f' + \mu^T g')(y, \eta(x, y)) < 0, \tag{27}$$

which contradicts dual constraint (20).

By condition (b), we have

$$(f' + \mu_{J_0} g'_{J_0} e)(y, \eta(x, y)) < 0,$$

and

$$\mu_{J_t} g'_{J_t}(y, \eta(x, y)) \leqslant 0, \quad \forall 1 \leqslant t \leqslant r.$$

Since $\xi \geqslant 0$, the above two inequalities yield

$$\left(\xi^T f' + \sum_{t=0}^{r} \mu_{J_t} g'_{J_t} \right)(y, \eta(x, y)) < 0.$$

The above inequality leads to (27), which contradicts (20).
By condition (c), we get

$$(f' + \mu_{J_0} g'_{J_0} e)(y, \eta(x, y)) < 0,$$

and

$$\mu_{J_t} g'_{J_t}(y, \eta(x, y)) < 0, \quad \forall 1 \leqslant t \leqslant r.$$

Since $\xi \geqslant 0$, the above two inequalities yield

$$\left(\xi^T f' + \sum_{t=0}^{r} \mu_{J_t} g'_{J_t} \right)(y, \eta(x, y)) < 0.$$

The above inequality leads to (27), which contradicts (20). This completes the proof. □

THEOREM 5.2 (Strong duality). *Let $\bar{x}$ be a locally weak Pareto efficient solution for* (P) *at which the generalized Slater's constraint qualification is satisfied, f and g be directionally differentiable at $\bar{x}$ with $f'(\bar{x}, \eta(x, \bar{x}))$, and $g'(\bar{x}, \eta(x, \bar{x}))$ preinvex functions on X and g_j be continuous for $j \in \hat{J}(\bar{x})$. Then, there exists $\bar{\mu} \in R_+^m$ such that $(\bar{x}, 1, \bar{\mu})$ is feasible for* (GMWD). *Moreover, if the weak duality between* (P) *and* (GMWD) *in Theorem* 5.1 *holds, then $(\bar{x}, 1, \bar{\mu})$ is a locally weak Pareto efficient solution for* (GMWD).

Proof. The proof of this theorem is similar to the proof of Theorem 4.2. □

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