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Simultaneous identification of diffusion coefficient, spacewise dependent source and initial value for one-dimensional heat equation

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This paper deals with an inverse problem of determining the diffusion coefficient, spacewise dependent source term, and the initial value simultaneously for a one-dimensional heat equation based on the boundary control, boundary measurement, and temperature distribution at a given single instant in time. By a Dirichlet series representation for the boundary observation, the identification of the diffusion coefficient and initial value can be transformed into a spectral estimation problem of an exponential series with measurement error, which is solved by the matrix pencil method. For the identification of the source term, a finite difference approximation method in conjunction with the truncated singular value decomposition is adopted, where the regularization parameter is determined by the generalized cross-validation criterion. Numerical simulations are performed to verify the result of the proposed algorithm. Copyright © 2016 John Wiley & Sons, Ltd.

Keywords: inverse problem; matrix pencil method; finite difference method; truncated singular value decomposition; generalized cross-validation

1. Introduction

Many transportation and diffusion processes in various scientific and engineering fields can be described by the following parabolic partial differential equation:

$$u_t(x,t) - D\Delta u(x,t) = f(x,t), \ (x,t) \in \Omega \times (0, t_{max}], \tag{1.1}$$

where $u(x,t)$ represents the state variable, D is the diffusion coefficient, and $f(x,t)$ the source term. In many industrial and engineering applications, however, some of the physical coefficients, surface heat flux or temperature history may be difficult or impossible to measure directly. It therefore becomes necessary to develop some algorithms and techniques to identify them from some other measurable information. These are the so called inverse heat conduction problems (IHCPs). Examples can be found in modeling of water contamination processes or air pollution phenomena, where an accurate estimation and control of the pollutant source $f(x,t)$ is crucial for environmental safeguard and protection [1, 2]. Another IHCP that occurs frequently in industry and combustion theory is the determination of heat flux and temperature on an inaccessible surface of a wall by measuring the temperature on an accessible boundary, for example, [3–5] and the references therein.

In the past decades, various classes of IHCPs have been investigated by many researchers ranging from estimation of thermal conductivity coefficients [6, 7], recovery of spatial distribution of heat sources [8, 9] and reconstruction of initial temperature distributions [10, 11]. Also, a large number of numerical methods in conjunction with some regularization techniques have been developed to solve IHCPs. These include the finite difference method (FDM) [8, 12], the finite element method [11, 13], the boundary element method [10], the method of fundamental solutions [9, 14, 15]. For other aspects including numerical solutions of inverse problems for partial differential equations, we refer to the monographs [16] and [17].

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It should be noted that most of the literature aforementioned consider the identification of only one unknown parameter or function. As is usually the case in many practical situations, however, one wishes to simultaneously reconstruct more than one physical quantities from some additional measured data, which becomes very complicated. To the best of our knowledge, papers devoted to this subject are very limited. The inverse problem of identifying both spacewise dependent source term and initial value from supplementary temperature measurements at two different instants of time is studied in [10], therein the iterative regularization algorithm is proposed and the boundary element method is used at each iteration. The same problem is solved in [18] and [15] by the regularized optimization method and the method of fundamental solutions, respectively. In [19], an inverse problem of determining a time-dependent heat source and initial temperature by means of observations of the temperature at the final time and temperature profile at one fixed point over the time interval is considered, where the numerical solutions are obtained by solving a backward heat conduction problem and two numerical derivative problems.

In comparison with the aforementioned works on identification of heat source that depends only on space or time, it is worth noting that the simultaneous reconstruction of the space-time dependent heat source and unknown initial temperature is studied in [20, 21], where the reconstruction is based on the temperature data at terminal time and the boundary observation data. This is different from that presented in [10, 15, 18]. For simultaneous identification of the initial temperature and the radiative coefficient for a heat conductive system, [11] investigates the stability and a numerical reconstruction algorithm is established based on the measurement of temperature at a fixed positive time and the temperature distribution in a subregion of the physical domain.

It can be seen that all the aforementioned literature are devoted to identification of one or two unknown parameters or functions, and few studies have dealt with the inverse problem that contains more than two unknowns. In this paper, we are concerned with a more general inverse problem that contains three unknown quantities, namely simultaneous identification of the constant diffusion coefficient, spacewise dependent source term and initial value for a one-dimensional heat equation, which is described by the following equation:

$$\begin{cases} w_t(x,t) - \alpha w_{xx}(x,t) = f(x), & 0 < x < 1,\ t > 0, \\ \alpha w_x(0,t) = u(t), \quad w(1,t) = 0,\ t \geq 0, & \\ w(x,0) = w_0(x), & 0 \leq x \leq 1, \end{cases} \tag{1.2}$$

with overspecified temperature measurement at time $T_1 > 0$,

$$\psi(x) = w(x, T_1),\ x \in [0,1], \tag{1.3}$$

and additional boundary measurement data

$$y(t) = w(0,t),\ t \in [0, T_2]. \tag{1.4}$$

In (1.2), $\alpha > 0$ is the diffusion coefficient, $f \in L^2[0,1]$ is the source term, $w_0 \in L^2[0,1]$ is the initial temperature distribution, and all of them are supposed to be unknown and need to be determined. The function $u(t)$ is the Neumann boundary control that represents the heat flux on the left boundary point $x = 0$, $\psi(x)$ is the measurement of temperature distribution at a fixed time $T_1 > 0$, and $y(t)$ the measurement of the temperature at the left end $x = 0$. Sometimes, we write the solution of (1.2) as $w = w(x,t;u,f,w_0)$ to denote its dependence on $u(t)$, $f(x)$ and $w_0(x)$.

In this paper, the inverse problem that we are concerned with can be described as follows:

Inverse Problem: *Reconstruct α, $f(x)$ and $w_0(x)$ simultaneously from the boundary control $u(t)$ and the additional measurement data $\{y(t), \psi(x)\}$.*

The rest of the paper is organized as follows. In Section 2, some preliminary results about the direct problem are presented. Section 3 is devoted to simultaneous identifiability analysis of the diffusion coefficient, source term and initial value. The identification algorithm is proposed in Section 4, where identification of the diffusion coefficient and initial value is mainly based on the matrix pencil method, and reconstruction of the source term is based on the finite difference approximation method in conjunction with the truncated singular value decomposition. In Section 5, a numerical example is presented to show the validity of the algorithm. Finally, concluding remarks are summarized in Section 6.

2. The direct problem

Before proceeding with analysis of the inverse problem, we present some preliminary results about the solution of the direct problem (1.2). Let $\mathcal{H} = L^2(0,1)$ be equipped with the usual inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Define the operator $\mathcal{A}:\ D(\mathcal{A})(\subset \mathcal{H}) \mapsto \mathcal{H}$ by

$$\begin{cases} (\mathcal{A}\phi)(x) = -\alpha \phi''(x), \\ D(\mathcal{A}) = \{\phi \in H^2(0,1) \,|\, \phi'(0) = \phi(1) = 0\}. \end{cases} \tag{2.1}$$

Proposition 2.1
For any $u \in L^2_{loc}(0,\infty)$, $w_0, f \in \mathcal{H}$, there exists a unique solution $w(x,t)$ to (1.2) such that $w \in C(0,\infty;\mathcal{H})$. In particular, $\psi(x) = w(x,T_1) \in \mathcal{H}$ for any $T_1 > 0$. In addition, $y(t) = w(0,t)$ is continuous with respect to $t > 0$.

Proof

It is well known that the operator $\mathcal{A}$ defined by (2.1) is positive definite and $-\mathcal{A}$ generates an analytic C_0-semigroup $e^{-\mathcal{A}t}$ on $\mathcal{H}$:

$$e^{-\mathcal{A}t}\tilde{w} = \sum_{n=1}^{\infty} \langle \tilde{w}, \phi_n \rangle e^{-\lambda_n t} \phi_n, \forall\ \tilde{w} \in \mathcal{H},$$

where $\{(\lambda_n, \phi_n(x))\}_{n\in\mathbb{N}}$ are eigen-pairs of $\mathcal{A}$ given by

$$\lambda_n = \alpha \left(n - \frac{1}{2}\right)^2 \pi^2,\ \phi_n(x) = \sqrt{2}\cos\left(n - \frac{1}{2}\right)\pi x,\ n = 1, 2, \ldots \tag{2.2}$$

Obviously, $\{\phi_n(x)\}_{n\in\mathbb{N}}$ forms an orthonormal basis for $\mathcal{H}$. Define operator $\mathcal{B} = -\delta(x)$, where $\delta(x)$ is the Dirac delta function. Then system (1.2) can be written as an evolutionary equation in $\mathcal{H}$:

$$\dot{w}(\cdot, t) = -\mathcal{A}w(\cdot, t) + \mathcal{B}u(t) + f(\cdot). \tag{2.3}$$

It is obvious that $\mathcal{B}\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{H})$ and $\mathcal{B}^*\tilde{w} = -\tilde{w}(0)$ for any $\tilde{w} \in D(\mathcal{A})$, and hence

$$\mathcal{B}^* e^{-\mathcal{A}^* t}\tilde{w} = -\sum_{n=1}^{\infty} \langle \tilde{w}, \phi_n \rangle e^{-\lambda_n t} \phi_n(0) = -\sqrt{2}\sum_{n=1}^{\infty} \langle \tilde{w}, \phi_n \rangle e^{-\lambda_n t}, \forall\ \tilde{w} \in D(\mathcal{A}),$$

So

$$\int_0^{\infty} \left|\mathcal{B}^* e^{-\mathcal{A}^* t}\tilde{w}\right|^2 dt \le \sum_{n=1}^{\infty} |\langle \tilde{w}, \phi_n \rangle|^2 \sum_{n=1}^{\infty} \lambda_n^{-1} = \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n}\right) \|\tilde{w}\|^2, \forall\ \tilde{w} \in D(\mathcal{A}).$$

This shows that $\mathcal{B}$ is admissible for $e^{-\mathcal{A}t}$ [23]. Therefore, for any $u \in L^2_{loc}(0, \infty)$, $w_0, f \in \mathcal{H}$, there exists a unique solution $w(x, t)$ to (1.2) such that $w \in C(0, \infty; \mathcal{H})$. The left is the continuity of $y(t)$.

Because $\{\phi_n(x)\}_{n\in\mathbb{N}}$ forms an orthonormal basis for $\mathcal{H}$, we express the solution $w(x, t)$ as

$$\begin{cases} w(x, t) = \sum\limits_{n=1}^{\infty} a_n(t)\phi_n(x), \\ a_n(t) = \int_0^1 w(x, t)\phi_n(x)dx. \end{cases} \tag{2.4}$$

Differentiate $a_n(t)$ with respect to t to obtain

$$a_n'(t) = \int_0^1 w_t(x, t)\phi_n(x)dx = -\sqrt{2}u(t) - \lambda_n a_n(t) + \langle f, \phi_n \rangle. \tag{2.5}$$

This together with the initial conditions $a_n(0) = \langle w_0, \phi_n \rangle$ gives

$$a_n(t) = \langle w_0, \phi_n \rangle e^{-\lambda_n t} + \int_0^t e^{-\lambda_n(t-s)} \left[\langle f, \phi_n \rangle - \sqrt{2}u(s)\right] ds. \tag{2.6}$$

Setting

$$\begin{aligned} G(t, x, y) &= -\sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x)\phi_n(y), \\ A_0(x) &= \sum_{n=1}^{\infty} \frac{\langle f, \phi_n \rangle}{\lambda_n} \phi_n(x), \quad \lambda_0 = 0, \\ A_n(x) &= \left[\langle w_0, \phi_n \rangle - \frac{\langle f, \phi_n \rangle}{\lambda_n}\right] \phi_n(x),\ n \in \mathbb{N}, \end{aligned} \tag{2.7}$$

the solution of system (1.2) can be represented as

$$w(x, t; u, f, w_0) = \sum_{n=0}^{\infty} A_n(x) e^{-\lambda_n t} + \int_0^t G(t-s, x, 0)u(s)ds,\ \forall\ x \in [0, 1],\ t \ge 0. \tag{2.8}$$

In particular, the boundary observation $y(t)$ is

$$y(t) = w(0, t; u, f, w_0) = \sum_{n=0}^{\infty} A_n(0) e^{-\lambda_n t} + \int_0^t G(t-s, 0, 0)u(s)ds,\ t > 0. \tag{2.9}$$

The first term on the right hand side of (2.9) is a uniformly convergent series for $t \ge T_0$, where $T_0 > 0$ can be arbitrarily small, hence it is continuous with respect to $t > 0$. Finally, because by (2.7), $G(t, 0, 0) = -2\sum_{n=1}^{\infty} e^{-\lambda_n t}$ is also continuous in $t > 0$, the second term on the right hand side of (2.9) is continuous with respect to t. So $y(t)$ is a continuous function over $t \in (0, \infty)$. This completes the proof the proposition. □

3. Identifiability

This section is devoted to identifiability analysis, that is, to establish if the data $\{u(t), y(t), \psi(x)\}$ is sufficient to simultaneously determine α, $f(x)$, and $w_0(x)$ uniquely, and vice versa in some extent that they are least information to determine these unknowns.

Obviously, the boundary observation $y(t)$ in (2.9) can be separated into two parts $w(0,t;0,f,w_0)$ and $w(0,t;u,0,0)$, where

$$w(0,t;0,f,w_0) = \sum_{n=0}^{\infty} A_n(0)e^{-\lambda_n t}, \tag{3.1}$$

is determined by the initial state $w_0(x)$ and source term $f(x)$, and

$$w(0,t;u,0,0) = \int_0^t G(t-s,0,0)u(s)ds, \tag{3.2}$$

is determined by the control $u(t)$. Moreover, the first part $w(0,t;0,f,w_0)$ is a Dirichlet series which implies that it can be determined by its restriction on any finite interval. Therefore, a natural idea for identification is first to estimate the unknown coefficients $\{(A_n(0),\lambda_n)\}_{n\in\mathbb{N}_0}$ in series (3.1), and whereafter, the part $w(0,t;0,f,w_0)$ can be canceled from the output $y(t)$. This transforms equivalently identification of diffusion coefficient free of initial value and source term.

In order to estimate $\{(A_n(0),\lambda_n)\}_{n\in\mathbb{N}_0}$ from the boundary observation $y(t)$, we let the boundary control $u(t) = 0$ during the time interval $t \in [0,T_1]$, where $T_1 > 0$ is any given time. This together with (3.2) shows that $w(0,t;u,0,0) = 0,\ t \in [0,T_1]$, and the measured output data $y(t)$ has the following expression:

$$y(t) \triangleq w(0,t;0,f,w_0) = \sum_{n=0}^{\infty} A_n(0)e^{-\lambda_n t},\ t \in (0,T_1]. \tag{3.3}$$

From uniqueness of the Dirichlet series, it is easy to show that $\{(A_n(0),\lambda_n)\}_{n\in\mathbb{N}_0}$ can be uniquely determined by $\{y(t), t\in[0,T_1]\}$. Then we let the boundary control $u(t) \neq 0$ for almost all $t \in (T_1,T_2]$, where $T_1 < T_2$, and the identifiability of the diffusion coefficient α can be achieved from the input–output mapping

$$\Phi:\ u(t) \mapsto w(0,t;u,0,0),\ t \in (T_1,T_2], \tag{3.4}$$

where $\{w(0,t;u,0,0), t \in (T_1,T_2]\}$ can be obtained from the output $\{y(t), t \in (T_1,T_2]\}$ by canceling the part $w(0,t;0,f,w_0)$, that is,

$$w(0,t;u,0,0) = y(t) - w(0,t;0,f,w_0) = y(t) - \sum_{n=0}^{\infty} A_n(0)e^{-\lambda_n t},\ t \in (T_1,T_2].$$

This is the basic idea for identifiability of the diffusion coefficient. Similarly with Theorems 2.1 and 2.2 of [24], we have Lemmas 3.1 and 3.2 which present identifiability of the diffusion coefficient.

Lemma 3.1
Suppose that $w_0, f \in \mathcal{H}$, and let λ_n and $A_n(x)$ be as defined in (2.2) and (2.7), respectively. Let the boundary control $u(t) = 0$ in $t \in [0,T_1]$ in system (1.2) for given $T_1 > 0$. Then the set $\{(A_k(0),\lambda_k) \mid A_k(0) \neq 0\}_{k\in\mathbb{N}_0}$ in (3.3) can be uniquely determined by the boundary observation data $\{y(t) = w(0,t;0,f,w_0) |\ t \in [0,T_1]\}$.

Proof
For $t \in [0,T_1]$, it is deduced from (2.9) that the boundary observation is (3.3). Because both $w_0(x)$ and $f(x)$ are unknown, it is not clear whether $A_n(0) \neq 0$ for any $n \in \mathbb{N}_0$. Define the set $\mathbb{K} \subset \mathbb{N}_0$ that is unknown as well and satisfies

$$A_k(0) \neq 0,\ k \in \mathbb{K};\ A_k(0) = 0,\ k \notin \mathbb{K}. \tag{3.5}$$

The proof is accomplished by two steps.

Step 1: Apply the Laplace transform to (3.3) to obtain

$$\hat{y}(s) = \sum_{n=0}^{\infty} \frac{A_n(0)}{s+\lambda_n} = \sum_{k\in\mathbb{K}} \frac{A_k(0)}{s+\lambda_k}, \tag{3.6}$$

where ˆ denotes the Laplace transform. Notice that we are only interested in those $A_k(0) \neq 0$. Hence, there is no zero/pole cancelations. In other words, $-\lambda_k$ is a pole of $\hat{y}(s)$ and $A_k(0)$ is the residue of $\hat{y}(s)$ at $-\lambda_k$ for any $k \in \mathbb{K}$. By the uniqueness of the Laplace transform, $\{(A_k(0),\lambda_k)\}_{k\in\mathbb{K}}$ is uniquely determined by $\{y(t) |\ t \in (0,\infty)\}$.

Step 2: The observation $y(t)$ in (3.3) for all $t > 0$ can be uniquely determined by its restriction on $I_1 = [0,T_1]$ because $y(t)$ is an analytic function in $t > 0$. This completes the proof of the lemma.

□

Lemma 3.2
Let $0 < T_1 < T_2 < \infty$, $w_0, f \in \mathcal{H}$ and let λ_n and $A_n(x)$ be as defined in (2.2) and (2.7), respectively. The control signal is chosen to be

$$u(t) = \begin{cases} 0, & t \in [0, T_1], \\ 1, & t \in (T_1, T_2], \end{cases} \tag{3.7}$$

and the corresponding boundary observation data is $\{y(t) = w(0, t; u, f, w_0) | \ t \in [0, T_2]\}$. Then the diffusion coefficient α in system (1.2) and the set

$$\left\{ \sum_{n=1}^{\infty} \frac{\langle f, \phi_n \rangle}{\lambda_n} \right\} \bigcup \left\{ A_n(0) = \sqrt{2} \left(\langle w_0, \phi_n \rangle - \frac{\langle f, \phi_n \rangle}{\lambda_n} \right) \right\}_{n \in \mathbb{N}} \tag{3.8}$$

can be uniquely determined by the observation $\{y(t) = w(0, t; u, f, w_0) | \ t \in [0, T_2]\}$.

Proof
By (2.9), it follows that

$$\tilde{y}(t) \triangleq y(t + T_1) - \sum_{k \in \mathbb{K}} A_k(0) e^{-\lambda_k (t + T_1)} = \int_0^t G(t - s, 0, 0) u(s + T_1) \, ds, \ t \in (0, T_2 - T_1]. \tag{3.9}$$

Because $u(t) \neq 0$ for $t \in (T_1, T_2]$, it follows from [23, Theorem 151] that $\{G(t, 0, 0) \ | \ t \in (0, T_2 - T_1]\}$ can be uniquely determined by $\{\tilde{y}(t) \ | \ t \in (0, T_2 - T_1]\}$. By Lemma 3.1, $\{(A_k(0), \lambda_k)\}_{k \in \mathbb{K}}$ can be uniquely determined from $\{y(t) \ | \ t \in [0, T_1]\}$, which shows, from the first equality of (3.9), that $\{\tilde{y}(t) \ | \ t \in (0, T_2 - T_1]\}$ can be obtained from $\{y(t) \ | \ t \in [0, T_2]\}$. Notice that for $t \in (0, T_2 - T_1]$, $G(t, 0, 0) = -2 \sum_{n=1}^{\infty} e^{-\lambda_n t}$. By Lemma 3.1 again, $\{\lambda_n\}_{n \in \mathbb{N}}$ can be uniquely determined by $\{G(t, 0, 0) \ | \ t \in (0, T_2 - T_1]\}$. The exponents $\{\lambda_n\}_{n \in \mathbb{N}}$ are therefore uniquely determined by $\{y(t) \ | \ t \in [0, T_2]\}$, and then the diffusion coefficient α can be obtained from (2.2). Having determined the diffusion coefficient α, the identifiability of the set $\{A_n(0)\}_{n \in \mathbb{N}_0}$ is attributed to the uniqueness of Dirichlet series expansion. This ends the proof of the lemma. □

Remark 3.1
It is clear from (3.8) in Lemma 3.2 that if one of the initial value $w_0(x)$ and source term $f(x)$ is known, the Fourier coefficients of the other one can be uniquely determined. For example, suppose that the source term $f(x)$ is known, then the diffusion coefficient α and initial value $w_0(x)$ can be simultaneously identified from $\{y(t) | \ t \in [0, T_2]\}$, which is reduced to the special case without source term in [24].

Remark 3.2
Observe that the Fourier coefficients of the initial value and the source term are generally mixed together in (3.8) of Lemma 3.2. So we can not distinguish $\{\langle f, \phi_n \rangle, \ \langle w_0, \phi_n \rangle\}$ from (3.8) directly. Actually, the knowledge of only one boundary measurement $y(t) = w(0, t)$ is not sufficient to uniquely determine the initial value and source term simultaneously, which is illustrated in Example 3.1.

Example 3.1
Let $\alpha = \alpha^* = 1$ in system (1.2). Consider the following two cases:
Case 1: Suppose that both the initial value and source term are zero, that is,

$$w_0(x) = f(x) = 0. \tag{3.10}$$

In this case, it is obvious that the boundary measurement $y(t)$ under the control $u(t)$ is

$$y(t) = w(0, t; u, 0, 0) = \int_0^t G(t - s, 0, 0) u(s) ds, \ t \geq 0. \tag{3.11}$$

Case 2: Suppose that the initial value and source term in system (1.2) are

$$\begin{aligned} \tilde{w}_0(x) &= \tfrac{4}{\pi^2} \left(\phi_1(x) - \phi_2(x) \right) = \tfrac{4\sqrt{2}}{\pi^2} \left(\cos \tfrac{1}{2} \pi x - \cos \tfrac{3}{2} \pi x \right), \ x \in [0, 1], \\ \tilde{f}(x) &= \phi_1(x) - 9 \phi_2(x) = \sqrt{2} \cos \tfrac{1}{2} \pi x - 9 \sqrt{2} \cos \tfrac{3}{2} \pi x, \quad x \in [0, 1]. \end{aligned} \tag{3.12}$$

In this case, it is easy to deduce from (2.7) that $A_n(0) = 0$ for all $n \in \mathbb{N}_0$. In view of (2.9), the boundary measurement $\tilde{y}(t)$ under the control $u(t)$ is also

$$\tilde{y}(t) = w(0, t; u, \tilde{w}_0, \tilde{f}) = \int_0^t G(t - s, 0, 0) u(s) ds, \ t \geq 0. \tag{3.13}$$

By comparing (3.11) with (3.13), it is seen that under the same control $u(t)$, different initial values and source terms produce the same boundary measurement $y(t) = \tilde{y}(t)$. In other words, the boundary observation data $y(t)$ does not contain enough information to determine the initial value and source term uniquely. Actually, a simple analysis as that in Example 3.1 can show that it is impossible to determine $w_0(x)$ and $f(x)$ uniquely from boundary temperature measurement or any other point measurement alone. That is why we choose (1.3) as additional measurement data, which can be obtained through interpolations of the point observation values in practical engineering applications.

The succeeding Theorem 3.1 indicates that α, $w_0(x)$, and $f(x)$ can be uniquely determined by the boundary control $u(t)$ and measurement data $\{y(t), \ \psi(x)\}$.

Theorem 3.1
Let $0 < T_1 < T_2 < \infty$, $w_0, f \in L^2(0,1)$ and let λ_n and $A_n(x)$ be as defined in (2.2) and (2.7), respectively. The control function $u(t)$ is chosen to be as that in (3.7) with the corresponding boundary observation data $\{y(t) = w(0,t;u,f,w_0)|\ t \in [0,T_2]\}$ and additional temperature distribution $\psi(x) = w(x,T)$ at time $t = T > 0$. Then $\{\alpha, w_0(x), f(x)\}$ in system (1.2) can be uniquely determined by the observation $\{y(t), \psi(x)\}$.

Proof
By (2.8), it follows that

$$\psi(x) = w(x,T;u,f,w_0) = \sum_{n=0}^{\infty} A_n(x)e^{-\lambda_n T} + \int_0^T G(T-s,x,0)u(s)ds,\ \forall\ x \in [0,1]. \tag{3.14}$$

According to the different values of T, there are three cases for the integral in (3.14):

Case 1: $0 < T \le T_1$. In this case, the integral in (3.14) is zero because $u(t) = 0,\ t \in [0,T_1]$.

Case 2: $T_1 < T \le T_2$. By (2.7), we have

$$\begin{aligned}\int_0^T G(T-s,x,0)u(s)ds &= -\sqrt{2}\sum_{n=1}^{\infty}\left(\int_{T_1}^T e^{-\lambda_n(T-s)}ds\right)\phi_n(x)\\ &= -\sqrt{2}\sum_{n=1}^{\infty}\frac{1-e^{-\alpha(n-\frac{1}{2})^2\pi^2(T-T_1)}}{\alpha(n-\frac{1}{2})^2\pi^2}\phi_n(x).\end{aligned} \tag{3.15}$$

Case 3: $T > T_2$. In this case,

$$\int_0^T G(T-s,x,0)u(s)ds = -\sqrt{2}\sum_{n=1}^{\infty}\frac{1-e^{-\alpha(n-\frac{1}{2})^2\pi^2(T_2-T_1)}}{\alpha(n-\frac{1}{2})^2\pi^2}\phi_n(x). \tag{3.16}$$

In each case, it is seen that the integral in (3.14) can be uniquely determined by the diffusion coefficient α. And by Lemma 3.2, it can also be uniquely determined by the boundary measurement $\{y(t)|\ t \in [0,T_2]\}$. Introduce

$$\tilde{\psi}(x) = \psi(x) - \int_0^T G(T-s,x,0)u(s)ds,\ \forall\ x \in [0,1]. \tag{3.17}$$

It is easy to see that $\tilde{\psi}(x)$ can be uniquely determined by $\{\psi(x), y(t)|\ t \in [0,T_2]\}$. On the other hand, it follows from (3.14) that

$$\tilde{\psi}(x) = \sum_{n=0}^{\infty} A_n(x)e^{-\lambda_n T} = \sum_{n=1}^{\infty}\left\{e^{-\lambda_n T}\left[\langle w_0,\phi_n\rangle - \frac{\langle f,\phi_n\rangle}{\lambda_n}\right] + \frac{\langle f,\phi_n\rangle}{\lambda_n}\right\}\phi_n(x). \tag{3.18}$$

Because $\{\phi_n(x)\}_{n\in\mathbb{N}}$ forms an orthonormal basis for $\mathcal{H}$, we have

$$e^{-\lambda_n T}\left[\langle w_0,\phi_n\rangle - \frac{\langle f,\phi_n\rangle}{\lambda_n}\right] + \frac{\langle f,\phi_n\rangle}{\lambda_n} = \langle\tilde{\psi},\phi_n\rangle,\ n \in \mathbb{N}. \tag{3.19}$$

Consequently,

$$\langle f,\phi_n\rangle = \lambda_n\langle\tilde{\psi},\phi_n\rangle - \lambda_n e^{-\lambda_n T}\left[\langle w_0,\phi_n\rangle - \frac{\langle f,\phi_n\rangle}{\lambda_n}\right],\ n \in \mathbb{N}. \tag{3.20}$$

By Lemma 3.2, the set $\left\{\langle w_0,\phi_n\rangle - \frac{\langle f,\phi_n\rangle}{\lambda_n}\right\}_{n\in\mathbb{N}}$ can be uniquely determined by $\{y(t)|\ t \in [0,T_2]\}$. This together with the fact that $\tilde{\psi}(x)$ can be uniquely determined by $\{\psi(x), y(t)|\ t \in [0,T_2]\}$ leads to that

$$f(x) = \sum_{n=1}^{\infty}\langle f,\phi_n\rangle\phi_n(x),\ x \in [0,1], \tag{3.21}$$

can be uniquely determined by $\{\psi(x), y(t)|\ t \in [0,T_2]\}$.

After the source term, $f(x)$, has been determined, the initial value can also be determined in terms of the analysis in Remark 3.1. This completes the proof of the theorem. □

Remark 3.3
It can be seen from Theorem 3.1 that the time T of the measured temperature distribution $w(x,T)$ is arbitrary and is not necessarily to be less than T_2. The choice of $T = T_1 < T_2$ in (1.3) is just for simplifying the numerical algorithms of estimating the initial state $w_0(x)$ and source $f(x)$ in Sections 4.2 and 4.3.

Remark 3.4
It should also be emphasized that the control signal (3.7) is crucial for simultaneous identifiability in Theorem 3.1, which is somehow like the persistently exciting condition in adaptive identification [25] although the method here is offline, and the role of (3.7) is to excite persistently the plant behavior. The following Example 3.2 indicates the necessity of the control signal (3.7), without which the simultaneous identifiability may not be valid anymore.

Example 3.2

Let $u(t) \equiv 0$ in system (1.2). Consider the following two cases:

Case 1: The diffusion coefficient α, initial value $w_0(x)$, and source term $f(x)$ are

$$\alpha = 1,\ w_0(x) = \frac{\phi_1(x)}{\lambda_1} = \frac{4\sqrt{2}}{\pi^2}\cos\frac{1}{2}\pi x,\ f(x) = \phi_1(x) = \sqrt{2}\cos\frac{1}{2}\pi x, \tag{3.22}$$

respectively.

Case 2: The diffusion coefficient, initial value, and source term are chosen to be

$$\tilde{\alpha} = 2\alpha,\ \tilde{w}_0(x) = w_0(x),\ \tilde{f}(x) = 2f(x), \tag{3.23}$$

respectively.

Simple calculations from (2.8) show that both cases produce the same temperature measurement data:

$$\begin{aligned} y(t) &= w(0,t;0,w_0,f) = w(0,t;0,\tilde{w}_0,\tilde{f}) = \frac{4\sqrt{2}}{\pi^2},\ t \geq 0, \\ \psi(x) &= w(x,T;0,w_0,f) = w(x,T;0,\tilde{w}_0,\tilde{f}) = \frac{4\sqrt{2}}{\pi^2}\cos\frac{1}{2}\pi x,\ x \in [0,1]. \end{aligned} \tag{3.24}$$

Hence, we can not distinguish $\{\alpha, w_0(x), f(x)\}$ and $\{\tilde{\alpha}, \tilde{w}_0(x), \tilde{f}(x)\}$ from the observation data $\{y(t), \psi(x)\}$ alone without the boundary control $u(t)$. Actually, in both cases, system (1.2) admits the same time-invariant steady-state solution

$$w(x,t) = \tilde{w}(x,t) = \frac{4\sqrt{2}}{\pi^2}\cos\frac{1}{2}\pi x,\ \forall\, x \in [0,1],\ t > 0. \tag{3.25}$$

That is why it is impossible to determine the unknown diffusion coefficient, initial value and source term uniquely from any measurement data.

To end this section, we point out that the inverse problem concerned in this paper is ill-posed. Actually, from the proof of Theorem 3.1, it is seen that both the identification of initial value $w_0(x)$ and source term $f(x)$ rely on the recovery of $\{A_n(0)\}$ from the measured data $y(t)$ in (3.3), which is known to be ill-posed. For example, if $w_0(x)$ and $f(x)$ satisfy that $A_n(0) \neq 0$ and $A_k(0) = 0,\ k \neq n$ for some $n \in \mathbb{N}$, then it follows from (3.3) that $y(t) = A_n(0)e^{-\lambda_n t}$, which implies that identification of $A_n(0)$ from $y(t)$ is ill-posed because the small error in the boundary measurement $y(t)$ can be enlarged by $e^{\lambda_n t} = e^{\alpha(n-\frac{1}{2})^2\pi^2 t}$ that can be very large if n is large. In addition, it is seen from (3.17) and (3.20) that the error in the measured data $\psi(x)$ can also be enlarged by $\lambda_n = \alpha(n-\frac{1}{2})^2\pi^2$ in the recovery of the Fourier coefficients of source term $f(x)$. Hence, some regularization methods are necessary in the identification algorithm in next section.

4. Identification algorithm

According to identifiability analysis in the previous section, identification algorithm of the diffusion coefficient, initial state and source term can be formulated into three steps, which are shown in the following subsections.

4.1. *Identification of the diffusion coefficient*

Suppose that $T_2 > T_1 > 0$, and the control signal $u(t)$ is chosen as in (3.7) with the corresponding boundary observation $\{y(t) = w(0,t;u,f,w_0) |\ t \in [0,T_2]\}$. Inspired by the identification algorithm in [24], identification of the diffusion coefficient α can be formulated into three steps based on the matrix pencil method (refer to e.g., [26, 27] for details):

Step 1: Estimate several eigenvalues of operator $\mathcal{A}$ from the boundary observation $\{y(t)|\ t \in [0,T_1]\}$ without control by the matrix pencil method.

Specifically, let $0 = t_0 < t_1 < \cdots < t_{N_1} = T_1$ be the equidistant sample points of $[0,T_1]$ with sampling period $T_s = \frac{T_1}{N_1}$, and the measured values at sample points are

$$\begin{aligned} y_i = \sum_{n=0}^{\infty} A_n(0)e^{-\lambda_n t_i} &= \sum_{k=0}^{M-1} A_{n_k}(0)e^{-(\lambda_{n_k}T_s)i} + \sum_{k=M}^{K-1} A_{n_k}(0)e^{-(\lambda_{n_k}T_s)i} \\ &\triangleq \sum_{k=0}^{M-1} A_{n_k}(0)e^{-(\lambda_{n_k}T_s)i} + e(M,i),\quad i = 0,1,\ldots,N_1-1, \end{aligned} \tag{4.1}$$

where the series $\{A_{n_k}(0)\}_{k=0}^{K-1}$ (K represents the number of the elements, which may be infinity) consists of all nonzero elements in the series $\{A_n(0)\}_{n\in\mathbb{N}_0}$. Then the number M of the estimable eigenvalues and the approximate eigenvalues $\left\{\tilde{\lambda}_{n_k}\right\}_{k=0}^{M-1}$ can be obtained by virtue of the matrix pencil method, where the remainder term $e(M,i)$ in (4.1) is treated as the measurement error.

Step 2: Estimate the coefficients $\{\tilde{A}_{n_k}(0)\}_{k=0}^{M-1}$ that correspond to $\left\{\tilde{\lambda}_{n_k}\right\}_{k=0}^{M-1}$ from (4.1) by solving the following linear least squares problem,

$$\{\tilde{A}_{n_k}(0)\}_{k=0}^{M-1} = \operatorname{argmin} \sum_{i=0}^{N_1-1} \left[y_i - \sum_{k=0}^{M-1} A_{n_k}(0) e^{-\tilde{\lambda}_{n_k} t_i} \right]^2 . \tag{4.2}$$

Step 3: Estimate the diffusion coefficient α through the boundary observation data $\{y(t) | \ t \in [T_1, \ T_2]\}$ by virtue of the matrix pencil method.

Similar to *Step 1*, let $T_1 = t_0' < t_1' < \cdots < t_{N_2}' = T_2$ be the uniform grids of $[T_1, T_2]$ with the sampling period $T_s' = \frac{T_2 - T_1}{N_2}$, and the control is chosen to be $u(t) = 1$ for $t \in [T_1, T_2]$. Then we have

$$\begin{aligned} y(t_i') &= w(0, t_i'; 0, f, w_0) + w(0, t_i'; u, 0, 0) \\ &= \sum_{n=0}^{\infty} A_n(0) e^{-\lambda_n t_i'} + \int_0^{t_i'} G(t_i' - s, 0, 0) u(s) ds \\ &\approx \sum_{k=0}^{M-1} \tilde{A}_{n_k}(0) e^{-\tilde{\lambda}_{n_k} t_i'} - \frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2}{\lambda_n} e^{-\lambda_n (t_i' - T_1)}. \end{aligned} \tag{4.3}$$

Let

$$y_i' = y(t_i') - \sum_{k=0}^{M-1} \tilde{A}_{n_k}(0) e^{-\tilde{\lambda}_{n_k} t_i'}, \ i = 0, 1, \ldots, N_2 - 1, \tag{4.4}$$

and

$$C_0' = -\frac{1}{\alpha}, \ \ \lambda_0' = 0, \ C_n' = \frac{2}{\lambda_n}, \ \ \lambda_n' = \lambda_n T_s', \ n \in \mathbb{N}. \tag{4.5}$$

Then (4.3) becomes

$$y_i' \approx \sum_{n=0}^{\infty} C_n' e^{-\lambda_n' i}, \ i = 0, 1, \ldots, N_2 - 1. \tag{4.6}$$

Next, estimate $\left\{(C_n', \lambda_n')\right\}_{n=0}^{M'-1}$ from (4.6) by repeating the processes in *Steps 1* and *2*. Then α can be obtained from (2.2) and (4.5).

Remark 4.1
It is easily seen that the algorithm for identification of the diffusion coefficient α here relies only on the boundary measurement data $y(t) = w(0, t)$ and has nothing to do with the distributed measurement data $\psi(x) = w(x, T_1)$. Here, we only list the brief steps for identification of the diffusion coefficient by virtue of the matrix pencil method, and the interested readers can refer to [24] for the details.

4.2. *Estimation of the initial value*

Now, we estimate the initial value. It is clear from (2.9) that

$$y(t) = \sum_{n=0}^{\infty} A_n(0) e^{-\lambda_n t} = A_0(0) + \sum_{n=1}^{\tilde{M}-1} A_n(0) e^{-\alpha (n - \frac{1}{2})^2 \pi^2 t} + e(\tilde{M}, t), \ \forall \ t \in (0, T_1], \tag{4.7}$$

where

$$e(\tilde{M}, t) = \sum_{n=\tilde{M}}^{\infty} A_n(0) e^{-\alpha (n - \frac{1}{2})^2 \pi^2 t}, \ \forall \ t \in (0, T_1]. \tag{4.8}$$

It is easy to verify that $\left|e(\tilde{M}, t)\right|$ will tend to zero as $\tilde{M} \to \infty$ (e.g., Theorem 3.1 of [24]). We can therefore choose an appropriate $\tilde{M}$ such that $\left|e(\tilde{M}, t)\right|$ is sufficiently small. Suppose that the observations at the sample points $0 = t_0 < t_1 < \cdots < t_N = T_1$ are available. The coefficients $\{A_n(0)\}$ can be estimated by solving the following minimization problem

$$\min \sum_{i=0}^{N-1} \left[y(t_i) - A_0(0) - \sum_{n=1}^{\tilde{M}-1} A_n(0) e^{-\alpha (n - \frac{1}{2})^2 \pi^2 t_i} \right]^2 , \tag{4.9}$$

or equivalently, finding the least squares solution of the matrix equation

$$CX = b, \tag{4.10}$$

where C is an $N \times \tilde{M}$ matrix with the (i,j) entries:

$$C(i,1) = 1,\ C(i,j) = e^{-\alpha(j-\frac{3}{2})^2\pi^2 t_{i-1}},\ i = 1,2,\ldots,N,\ j = 2,3,\ldots,\tilde{M}, \tag{4.11}$$

and

$$X = [A_0(0), A_1(0), \ldots, A_{\tilde{M}-1}(0)]^\top,\ b = [y(t_0), y(t_1), \ldots, y(t_{N-1})]^\top. \tag{4.12}$$

Because the reconstruction of the initial value is known to be ill-posed, which results in the ill-conditioned matrix C in (4.10), the standard numerical methods can not achieve good accuracy in solving (4.10). In order to overcome the instability, here we use the truncated singular value decomposition (TSVD) [28]. Suppose that the SVD of matrix C is

$$C = U\Sigma V^\top, \tag{4.13}$$

where $U = [u_1, u_2, \ldots, u_N]$ and $V = [v_1, v_2, \ldots, v_{\tilde{M}}]$ are orthonormal matrices, and $\{u_i\}_{i=1}^N$ and $\{v_i\}_{i=1}^{\tilde{M}}$ are the left and right singular vectors, respectively. In addition, $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots)$ is a diagonal matrix with non-negative diagonal elements that are singular values of C. In the TSVD method, the matrix C is replaced by its rank-k approximation, and the regularized solution is given by

$$X_{reg} = \sum_{i=1}^{k} \frac{u_i^\top b}{\sigma_i} v_i \triangleq C^I b, \tag{4.14}$$

where $k \le \text{rank}(C)$ is the regularization parameter. In this paper, we use the generalized cross-validation (GCV) criterion [29] to determine the regularization parameter k. The GCV criterion determines the optimal regularization parameter k by minimizing the following GCV function:

$$G(k) = \frac{\|CX_{reg} - b\|^2}{(\text{trace}(I_N - CC^I))^2}. \tag{4.15}$$

Remark 4.2

Having obtained the regularized solution X_{reg}, we then obtain the approximate values of the set:

$$\left\{A_n(0) = \sqrt{2}\left(\langle w_0, \phi_n\rangle - \frac{\langle f, \phi_n\rangle}{\lambda_n}\right)\right\}_{n=1}^{\tilde{M}-1}. \tag{4.16}$$

It has been shown in Remark 3.2 and Example 3.1 that it is impossible to determine the initial value $w_0(x)$ and source term $f(x)$ uniquely from (4.16). We therefore need to find out some other relations between them by making use of the other measured data $\psi(x)$.

On the other hand, it follows from (1.3), (2.8), and (3.7) that

$$\psi(x) = \sum_{n=0}^{\infty} A_n(x) e^{-\lambda_n T_1} = \sum_{n=1}^{\infty} \left\{ e^{-\lambda_n T_1} \left[\langle w_0, \phi_n\rangle - \frac{\langle f, \phi_n\rangle}{\lambda_n} \right] + \frac{\langle f, \phi_n\rangle}{\lambda_n} \right\} \phi_n(x). \tag{4.17}$$

In view of the orthonormal basis property of $\{\phi_n(x)\}_{n\in\mathbb{N}}$ for $\mathcal{H}$, we have

$$e^{-\lambda_n T_1} \left[\langle w_0, \phi_n\rangle - \frac{\langle f, \phi_n\rangle}{\lambda_n} \right] + \frac{\langle f, \phi_n\rangle}{\lambda_n} = \langle \psi, \phi_n\rangle,\ n \in \mathbb{N}. \tag{4.18}$$

By (4.16) and (4.18),

$$\langle w_0, \phi_n\rangle = \frac{1 - e^{-\lambda_n T_1}}{\sqrt{2}} A_n(0) + \langle \psi, \phi_n\rangle,\ n = 1, 2, \ldots, \tilde{M} - 1. \tag{4.19}$$

The initial value can then be estimated via its asymptotic Fourier series expansion:

$$\begin{aligned} w_0(x) \approx \bar{w}_0(x) &= \sum_{n=1}^{\tilde{M}-1} \langle w_0, \phi_n\rangle \phi_n(x) \\ &= \sum_{n=1}^{\tilde{M}-1} \left[\left(1 - e^{-\lambda_n T_1}\right) A_n(0) + \sqrt{2}\langle \psi, \phi_n\rangle \right] \cos\left(n - \frac{1}{2}\right)\pi x. \end{aligned} \tag{4.20}$$

Remark 4.3

Actually, the Fourier coefficients of the source term $f(x)$ can also be obtained from (4.16) and (4.18), namely,

$$\langle f, \phi_n\rangle = \lambda_n \langle \psi, \phi_n\rangle - \frac{\lambda_n e^{-\lambda_n T_1}}{\sqrt{2}} A_n(0),\ n = 1, 2, \ldots, \tilde{M} - 1. \tag{4.21}$$

However, it is inevitable that there is some measurement error in the measured data $\psi(x)$. By the expression of λ_n in (2.2), it is obvious that the error of the Fourier coefficient $\langle f, \phi_n\rangle$ will be enlarged as n increases. Hence, it is not stable to identify the source term directly, which is in contrast to the estimation of the initial value in equation (4.20).

4.3. Recovery of the source term

In this section, inspired by [8], we use the finite difference approximation together with the TSVD to determine the source term $f(x)$ in (1.2). To begin with, divide $[0, T_1]$ into N equally spaced intervals of t with the time step size Δt, that is, $t_k = k\Delta t,\ k = 0, 1, \dots, N$, and divide $[0, 1]$ into m equally spaced intervals of x with the spatial step size Δx, that is, $x_i = i\Delta x,\ i = 0, 1, \dots, m$.

It is known that a good choice of FDM for heat equation is the Crank-Nicolson scheme, which is known for its unconditional stability and second-order accuracy in both Δt and Δx [30]. Denoting w_i^k as an approximate value of $w(x_i, t_k)$ and $f_i = f(x_i)$, the heat equation in (1.2) can be discretized using the Crank-Nicolson (θ weighted) method

$$\frac{w_i^{k+1} - w_i^k}{\Delta t} = \alpha\theta\left[\frac{w_{i-1}^{k+1} - 2w_i^{k+1} + w_{i+1}^{k+1}}{\Delta x^2}\right] + \alpha(1-\theta)\left[\frac{w_{i-1}^k - 2w_i^k + w_{i+1}^k}{\Delta x^2}\right] + f_i, \tag{4.22}$$

where $0 \le \theta \le 1,\ i = 1, 2, \dots, m-1,\ k = 0, 1, \dots, N-1$. For the boundary condition $w_x(0, t) = 0$, by introducing a ghost node w_{-1} and using the central difference approximation at $x = 0$ $(x = x_0)$, we can compute the value of w_{-1} by extrapolation, that is, $w_{-1} = w_1$. For $i = 0,\ k = 0, 1, \dots, N-1$, we have

$$\frac{w_0^{k+1} - w_0^k}{\Delta t} = 2\alpha\theta\left[\frac{w_1^{k+1} - w_0^{k+1}}{\Delta x^2}\right] + 2\alpha(1-\theta)\left[\frac{w_1^k - w_0^k}{\Delta x^2}\right] + f_0. \tag{4.23}$$

For the boundary condition $w(1, t) = 0$, we have

$$w_m^k = 0,\ k = 0, 1, \dots, N-1. \tag{4.24}$$

Introduce the vector

$$W^k = \left[w_0^k, w_1^k, \dots, w_m^k\right]^\top,\ k = 0, 1, \dots, N. \tag{4.25}$$

The discrete difference equations (4.22)–(4.24) can be transformed into the difference scheme in a matrix form as

$$H_+ W^{k+1} = H_- W^k + F,\ k = 0, 1, \dots, N-1, \tag{4.26}$$

where

$$H_+ = \begin{bmatrix} \beta_1 & 2\gamma_1 & 0 & 0 & \cdots & 0 \\ \gamma_1 & \beta_1 & \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_1 & \beta_1 & \gamma_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \gamma_1 & \beta_1 & \gamma_1 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix},\ H_- = \begin{bmatrix} \beta_2 & 2\gamma_2 & 0 & 0 & \cdots & 0 \\ \gamma_2 & \beta_2 & \gamma_2 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \beta_2 & \gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \gamma_2 & \beta_2 & \gamma_2 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.27}$$

and

$$F = [f_0, f_1, f_2, \dots, f_{m-1}, 0]^\top, \tag{4.28}$$

$$\beta_1 = \frac{1}{\Delta t} + \frac{2\alpha\theta}{\Delta x^2},\quad \gamma_1 = -\frac{\alpha\theta}{\Delta x^2},\quad \beta_2 = \frac{1}{\Delta t} - \frac{2\alpha(1-\theta)}{\Delta x^2},\quad \gamma_2 = \frac{\alpha(1-\theta)}{\Delta x^2}. \tag{4.29}$$

Left multiply with H_+^{-1}, the inverse of H_+, on both sides of (4.26) to yield

$$W^{k+1} = DW^k + H_+^{-1}F,\ k = 0, 1, \dots, N-1, \tag{4.30}$$

where $D = H_+^{-1}H_-$. We can then express the final temperature W^N at time $t = T_1$ as

$$W^N = D^{(N)}W^0 + BF, \tag{4.31}$$

where

$$D^{(i)} = D^{(i-1)}D,\ i = 1, 2, \dots, N,\quad D^{(0)} = I,\quad B = \sum_{i=0}^{N-1} D^{(i)}H_+^{-1}, \tag{4.32}$$

$$W^N = [\psi(x_0), \psi(x_1), \psi(x_2), \dots, \psi(x_m)]^\top.$$

Because we have identified the diffusion coefficient α in Section 4.1, by substituting it into (4.27) and (4.29), we obtain the matrices H_+ and H_- from which we can compute the matrices $D^{(N)}$ and B in (4.31) from (4.32). The initial temperature distribution vector W^0 can also be given by the estimation of the initial state, $\bar{w}_0(x)$, in Section 4.2 as

$$W^0 \approx [\bar{w}_0(x_0), \bar{w}_0(x_1), \bar{w}_0(x_2), \dots, \bar{w}_0(x_m)]^\top. \tag{4.33}$$

Table I. The identification algorithm.
Algorithm: Simultaneous reconstruction of diffusion coefficient, source and initial value
Step 1. Take $u(t) = 0,\ t \in [0, T_1]$ and estimate the approximate eigenvalues $\left\{\tilde{\lambda}_{n_k}\right\}_{k=0}^{M-1}$ from (4.1) by virtue of the matrix pencil method. **Step 2.** Estimate the coefficients $\{\tilde{A}_{n_k}(0)\}_{k=0}^{M-1}$ by solving (4.2). **Step 3.** Take $u(t) = 1,\ t \in (T_1, T_2]$ and estimate the diffusion coefficient α via matrix pencil method by (4.4)–(4.6). **Step 4.** Compute the set $\{A_n(0)\}_{n=1}^{\tilde{M}-1}$ in (4.16) by (4.10)–(4.14). **Step 5.** Reconstruct the initial data $w_0(x)$ by (4.19)–(4.20). **Step 6.** Choose the weight θ and positive integers $N,\ m$, and recover the heat source, f, by solving (4.35) with truncated singular value decomposition.

Introducing the following notation

$$\Phi = W^N - D^{(N)} W^0, \tag{4.34}$$

the identification of the source term, $f(x)$, is reduced to solving the following matrix equation

$$BF = \Phi. \tag{4.35}$$

Because the original inverse source problem is ill-posed that results in the ill-posedness of the matrix equation (4.35). Hence, we can not solve (4.35) directly and some regularization method is necessary. Similar to estimation of the initial state by solving (4.10), we use the TSVD once again to solve the matrix equation (4.35) arising from the FDM. Here, we omit the details of the TSVD.

4.4. *Identification algorithm*

To sum up, the algorithm for simultaneously reconstructing the diffusion coefficient, source and initial value can be formulated as Table I.

5. Numerical example

In this section, we present an example to demonstrate the effectiveness of the proposed algorithm, where the implementation of the algorithm is based on the MATLAB software, and the relevant coefficients are taken to be

$$T_1 = 0.5,\ T_2 = 1,\ \theta = \frac{1}{2},\ \Delta x = 0.01,\ \Delta t = 0.01.$$

In the computational process, the measurement vectors $Y_t = \{y(t_k)\}_{k=0}^{101}$ and $\Psi_x = \{\psi(x_i)\}_{i=0}^{101}$ are obtained actually at the points of the mesh grid. The former boundary measurement data Y_t is supposed to be almost accurate, while the latter measured data Ψ_x is supposed to be contaminated by some errors because of the inherent difficulty of distributed measurement in practice. Therefore, we replace the exact data Ψ_x by adding some random distributed perturbations, that is,

$$\tilde{\Psi}_x = \Psi_x(1 + \delta \cdot \text{rand}(\text{size}(\Psi_x))), \tag{5.1}$$

where the magnitude δ indicates the percentage error level, and rand$(\cdot)$ is a random number in [0, 1] that is realized by using the MATLAB function *rand*.

Example 5.1
The constant diffusion coefficient, initial condition, and spacewise dependent heat source in (1.2) are chosen as

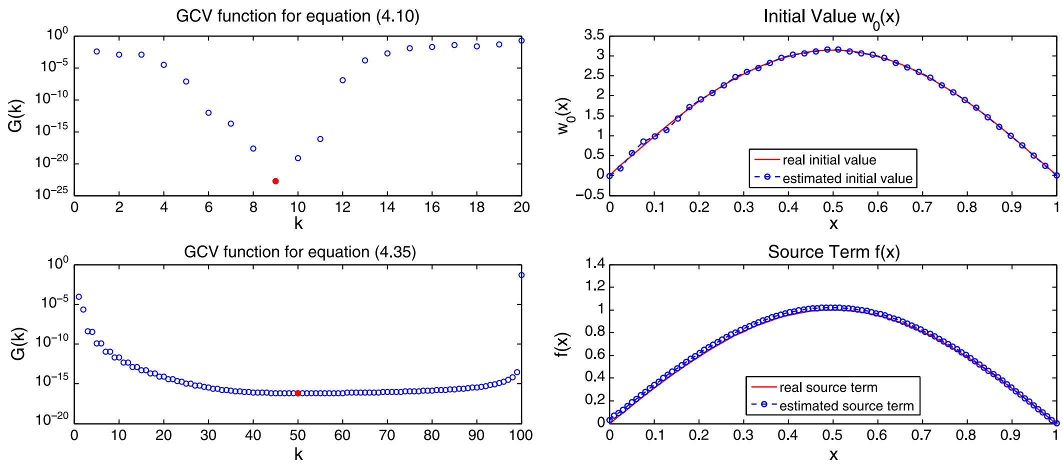
$$\alpha^* = 1,\ u_0^*(x) = \pi \sin \pi x,\ f^*(x) = \sin \pi x, \tag{5.2}$$

respectively. Although it is difficult to obtain the analytical solution of the direct problem (1.2), the measured output data Y_t and Ψ_x can be obtained from (2.8), where the infinite series is approximated by a finite one.

The estimated $\{C_n', \lambda_n'\}_{n=0}^{8}$ (4.5) are partially listed in Table II, where it can be seen that the diffusion coefficient is $\alpha^* \approx \tilde{\alpha} = 1.0000$. Figure 1(a) presents the GCV functions obtained for the inverse initial value problem and inverse source problem using the TSVD to solve the equations (4.10) and (4.35) without random noise in the measured output data. It can be seen from the figure that the minima of the GCV functions occur at $k = 9$ and $k = 50$ for (4.10) and (4.35), respectively. Similar results can be obtained for other cases in which random noise is added to the measured data, which are omitted here. The comparison between the exact initial value and source term and the numerical results are shown in Figure 1(b), where the exact solutions are given by the solid line and the numerical results are shown as the curve plotted with circles. The figure shows that the numerical results are quite satisfactory.

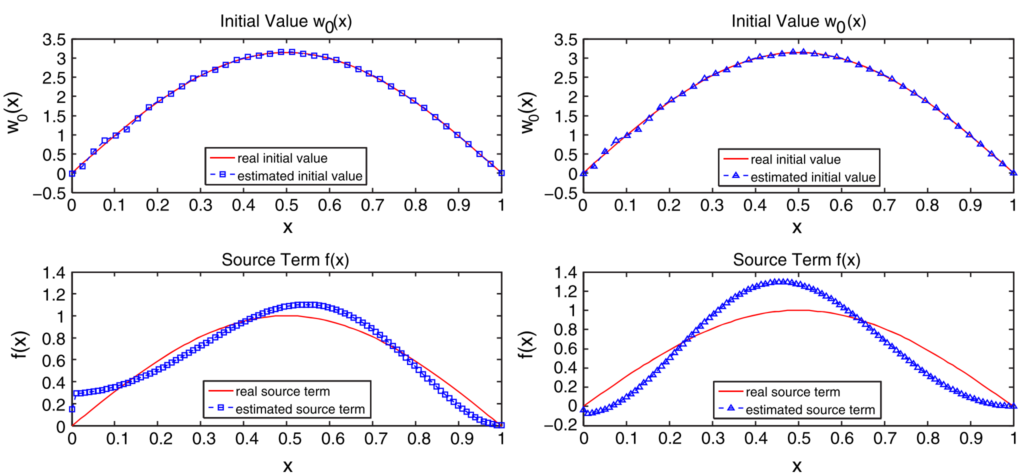
Table II. The estimated $\{C'_n, \lambda'_n\}_{n=0}^{6}$.

n	0	1	2	3	4	5	6
$100 * C'_n$	-131.8310	81.0569	9.0063	3.2423	1.6542	1.0007	0.6719
$100 * \lambda'_n$	0.0000	2.4674	22.2066	61.6850	120.9027	199.8596	298.6178
$100 * C'_n * \lambda'_n$	/	2.0000	2.0000	2.0000	2.0000	2.0000	2.0064
$\alpha^* \approx \frac{\lambda'_n}{T'_s(n-\frac{1}{2})^2\pi^2}$	/	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000



(a) the GCV function (b) the initial value and source term

Figure 1. The generalized cross-validation (GCV) analysis and reconstruction of initial value $w_0(x)$ and source term $f(x)$ using measured data without random error: (a) the GCV function; (b) initial value and source term. [Colour figure can be viewed at wileyonlinelibrary.com]



(a) the initial value and source term ($\delta = 1\%$) (b) the initial value and source term ($\delta = 3\%$)

Figure 2. The initial value $w_0(x)$ and source term $f(x)$ for various levels of noise contaminated measured data: (a) with 1% random error; (b) with 3% random error. [Colour figure can be viewed at wileyonlinelibrary.com]

The numerical results for the initial value and the corresponding heat source with various relative noise levels, $\delta = 1\%, \ 3\%$, are depicted in Figures 2(a) and 2(b), respectively. It can be seen that accurate estimation for the initial value can be made even for the distributed measured data with some errors. The reconstruction of the heat source is also acceptable, although there seems to be a slight disagreement with the exact one. One possible reason is that in the computation of the heat source, the required initial value is replaced by the reconstructed one. We can therefore say that not only the noise corrupted distributed measured data but also the reconstructed initial value have influence on the identification of the heat source.

6. Concluding remarks

In this paper, a numerical algorithm for simultaneous identification of the constant diffusion coefficient, initial value and spacewise dependent source term in a one-dimensional heat conduction equation is developed. It is shown that identification of the diffusion coefficient relies only on the boundary measurement which admits a Dirichlet series representation. The inverse coefficient problem is then transformed into a spectral estimation problem and is solved by the matrix pencil method. The initial value is approximated by its Fourier series expansion, and the source term is recovered by the FDM together with the truncated singular value decomposition method. Actually, after the diffusion coefficient has been identified, the numerical methods developed in other papers (e.g., [21]) are also applicable for the reconstruction of the initial value and source term. The method used in this paper is more direct and simple. Simulation examples have also been performed to confirm the effectiveness of the proposed algorithm.

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