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C_0P_2–P_0 Stokes finite element pair on sub-hexahedron tetrahedral grids

Shangyou Zhang[1] · **Shuo Zhang**[2]

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Abstract This paper presents a procedure to construct stable $C_0P_2-P_0$ finite element pair for three dimensional incompressible Stokes problem. It is proved that, the quadratic-constant finite element pair, though not stable in general, is uniformly stable on a certain family of tetrahedral grids, namely some kind of sub-hexahedron tetrahedral grids. The sub-hexahedron tetrahedral grid is defined by refining each eight-vertex hexahedron of a certain hexahedral grid into twelve tetrahedra with one added vertex inside the hexahedron, while the hexahedral grid is a partition of a polyhedral domain where each (non-flat face) hexahedron is defined by a tri-linear mapping on the unit cube with eight vertices.

Keywords Stokes problem · Mixed finite element · Continuous quadratic velocity · Discontinuous pressure · Hexahedral grid · Tetrahedral grid

1 Introduction

This paper studies the discretisation of the incompressible Stokes problem of velocity-pressure type

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✉ Shuo Zhang
szhang@lsec.cc.ac.cn

Shangyou Zhang
szhang@udel.edu

1 Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA

2 LSEC, ICMSEC and NCMIS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded polyhedral domain, $\mathbf{u}$ is the velocity and p is the pressure, of a flowing fluid. The variational form of the Stokes equations reads: Find $\mathbf{u} \in H_0^1(\Omega)^3$ and $p \in L_0^2(\Omega)(:= L^2/C)$ such that

$$\begin{aligned} (\nabla \mathbf{u}, \nabla \mathbf{v}) - (\operatorname{div} \mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \quad & \forall \mathbf{v} \in H_0^1(\Omega)^3, \\ (\operatorname{div} \mathbf{u}, q) = 0 \quad & \forall q \in L_0^2(\Omega). \end{aligned}$$

In this paper, we use the standard notation for Sobolev spaces.

In the conforming finite element method, an H_0^1-subspace $\mathbf{V}_h$ of continuous piecewise polynomials is chosen along with a proper L_0^2-subspace P_h of (continuous or discontinuous) piecewise polynomials, satisfying the inf-sup stability condition

$$\inf_{q_h \in P_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(\operatorname{div} \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{H^1} \|q_h\|_{L^2}} \geqslant \beta_0 > 0. \tag{2}$$

Provided the inf-sup condition, the solutions of the Stokes problem (1) can be approximated quasi-optimally [4,8,11] :

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1} + \|p - p_h\|_{L^2} \leq C \left(\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{H^1} + \inf_{p_h \in P_h} \|p - p_h\|_{L^2} \right), \tag{3}$$

where $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in P_h$ satisfy

$$\begin{cases} (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (\operatorname{div} \mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\operatorname{div} \mathbf{u}_h, q_h) = 0 & \forall q_h \in P_h. \end{cases} \tag{4}$$

The construction of a pair of finite element spaces that are conforming subspaces to the velocity space and pressure space, respectively, and that satisfies the condition (2) is a key issue in the numerical solution of the Stokes problem, and numerous manuscripts are devoted to that.

A most natural pair is the $C_0P_k - P_{k-1}$ element pair where the velocity space consists of continuous piecewise polynomials of degree k, while the pressure space is the discontinuous piecewise polynomials of degree $k-1$. Once the pair is inf-sup stable, the divergence-free condition of the velocity is rigorously preserved. Scott and Vogelius proved that the $C_0P_k - P_{k-1}$ finite element is inf-sup stable and quasi optimal on 2D triangular grids under very mild assumptions, if the polynomial degree $k \geq 4$, cf. [23,24]; while for $k < 4$ in 2D, the $C_0P_k - P_{k-1}$ finite element is not stable on general triangular grids. When three dimensional case is taken under consideration, for a general tetrahedral grid, it remains open if there is a magic number χ, such that $C_0P_k - P_{k-1}$ pair is inf-sup stable for $k \geq \chi$. The counterpart problem on rectangular grids, in 2D and in 3D, is completely solved [12,13,35,38]. Also, some partial solutions are derived on special tetrahedral grids. On the uniform tetrahedral grids (each cube splitted into six tetrahedra), the $C_0P_k-P_{k-1}$ element is inf-sup stable if $k \geq 6$ [36]. On the Hsieh-Clough-Tocher grids (each tetrahedron is refined into four tetrahedra by connecting its bary-center to the four vertices), the $C_0P_k-P_{k-1}$ element is inf-sup stable if $k \geq 3$ [32]. Further, on the Powell-Sabbin grids (each tetrahedron is refined into twelve tetrahedra by connecting its bary-center to the four vertices and the four face bary-centers), the $C_0P_k-P_{k-1}$ element is inf-sup stable if $k \geq 2$ [37].

It can be expected that using relatively smaller pressure space makes the inf-sup condition easier to be satisfied and deduces possibly low-degree stable pairs. The space for pressure can be reduced by enhancing the smoothness of the functions. The Taylor-Hood element, cf. [3,7,9,27,28], uses continuous space for pressure and obtains the $C_0P_k - C_0P_{k-1}$ stable pair. For pairs of such kind, the conservation of mass can not be guaranteed in any sub-region, whereas, in some situations, the mass-conservation is necessary, cf. [15–18,21,22]. One remedy is to enrich the pressure space by piecewise constant functions, [3,5,29,38], and we would thus look for stable pairs whose pressure space contains completely piecewise constants. Meanwhile, Falk and Neilan [10,19] reduces the pressure space to the piecewise P_{k-1} polynomials continuous at vertex in 2D, or C_1 continuous at 3D vertex and continuous at 3D edges, showed that such $C_0^+P_k - C_{-1}^-P_{k-1}$ elements are inf-sup stable for $k \geq 4$ in 2D and for $k \geq 6$ in 3D, where C_0^+ denotes functions C_1 at vertices in 2D, or functions C_2 at vertices and C_1 at edges in 3D. The mass conservation can be enforced pointwise by these elements.

Meanwhile, using a relatively bigger velocity space is also a natural idea. A most significant example along this line is the Bernardi–Raugel pair [2], which uses bubbles to enhance piecewise linear polynomials for velocity with the piecewise constant pressure. As only edge bubbles are added in, the pair confirms the P_2-P_0 pair in two dimension. However, when we try to repeat the procedure to construct the Bernardi–Raugel element in three dimension, we will find in contrast to the two dimensional case, face bubbles are needed in three dimension, and thus we can only obtain the stability of the P_3-P_0 pair. The applicability of $C_0P_2-P_0$ is still not known. Also, as the Taylor-Hood pair is constructed in three dimension, when we consider to enhance piecewise constant in the pressure space, cubic polynomials seem necessary for velocity space. It will be meaningful for us to consider if we can use a low-degree (say, quadratic) velocity space to generate a stable pair with a pressure space of piecewise constant.

In this paper, we present a detailed procedure to construct stable $C_0P_2-P_0$ finite element pair of the Stokes problem. We do not try to discuss the general validity of the $C_0P_2-P_0$ finite element pair; instead, we show that the $C_0P_2-P_0$ element is stable on certain sub-hexahedron tetrahedral grids, which can be generated in two steps below. First, we subdivide the domain Ω with $\mathcal{Q}_h$ a hexahedral grid on Ω where each hexahedron is defined by eight points and a trilinear mapping from the eight vertices of the unit cube to these eight points. Such a hexahedron may not have flat faces, i.e., four "face" points may not be on a same plane. There is no (standard) geometric definition of regularity of hexahedra yet. Here we assume [33] that the 32 tetrahedra, 24 formed by adding two points not on a same face to the two end points of an edge of the 12 edges and the other 8 formed by three edges at each of 8 vertices,

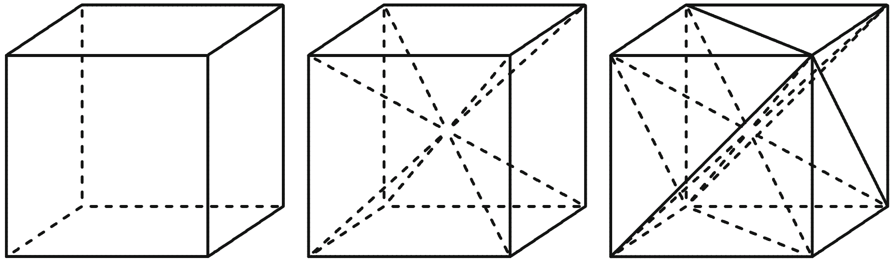

Fig. 1 A general hexahedron is split into 12 tetrahedra

are all shape regular (a bounded Jacobian matrix for the 32 affine mappings.) We note that the previous 24-tetrahedra test [14] does not guarantee shape regular of the hexahedral grid. Second, with this hexahedral grid $\mathcal{Q}_h$, we split each hexahedron into 12 tetrahedra by connecting its bary-center point with each of two face triangles on six faces, cf. Fig. 1, to get a tetrahedral grid $\mathcal{T}_h$ on Ω. The selection of a face triangle is not unique (two choices to split a quadrilateral into two triangles.) We will make the same choice on the two sides of a common face of neighboring hexahedra, to generate a conforming grid. We prove in this paper that the $C_0P_2-P_0$ element is stable on such "general" tetrahdral grids.

We note further, if we refine properly such a grid by splitting each tetrahedron into eight half-sized tetrahedra (unlike the 2D uniqueness, there are 3 ways to split a tetrahedron, cf. [31]), we would get another such sub-hexahedron tetrahedral grid. In other words, if we refine the hexahedral grid $\mathcal{Q}_h$ into a half-sized grid first, then we refine each hexahedron into 12 tetrahedra, the resulting tetrahedral grids would be the same. That is, the $C_0P_2-P_0$ element is actually stable on the family of refining multigrids. This is indeed a key difference between our method and some existing finite element pairs that are stable on specific structured grids. There are many examples in this category. In two dimension, the $C_0P_k-P_{k-1}$ finite element, though generally not, is still stable for small k on some structured grids, such as criss-cross grids, Hsieh-Clough-Tocher grids, Power-Sabbin grids, cf. [1,20,30,34]. The main concern of these methods is that the structures used can not been preserved under the nested refinement process.

This way, we arrive at the conclusion that, for domains that can be covered by the hexahedral grids, the P_2-P_0 pair can be inf-sup stable. This method is practically applicable, as the pair can preserve mass element-wise while reducing the polynomial degree from three to two. Further, the pair can then be used as analytic tools for other more complicated elements. For example, the HCT P_2/P_1 pair is also stable on such sub-hexahedron tetrahedral grids, cf. [32].

The rest of the paper is organized as follows. In Sect. 2, we define the grid and the finite element spaces. In Sect. 3, we prove the inf-sup stability for the $C_0P_2-P_0$ element and its optimal order of convergence. In Sect. 4, we provide a simple numerical test, verifying the theory.

2 The C_0P_2–P_0 finite element

Given eight ordered points $\mathbf{x}_i$ in 3D, cf, the center diagram in Fig. 4, there is a unique tri-linear mapping defined by

$$F = \sum_{i=1}^{8} \mathbf{x}_i \phi_i(\hat{\mathbf{x}}) : [0, 1]^3 \to K := F([0, 1]^3),$$

mapping the eight vertices of the unit cube to the eight points, where ϕ_i are the trilinear functions which evaluates 1 on $\mathbf{x}_i$ and vanishes on $\mathbf{x}_j$ for $1 \le j \ne i \le 8$. The image, namely $F([0, 1]^3)$, is called a hexahedron. We deal with shape-regular hexahedra K, but we allow non-convex hexahedra, for example, the first four points $\mathbf{x}_i$ are not on a same plane.

To define the shape-regularity of a hexahedron K, we introduce the linear (not trilinear) reference mapping $F_T : \hat{T} \to T$, cf. Fig. 2,

$$\begin{aligned} F_T &= \sum_{i=1}^{4} \mathbf{x}_i \lambda_i(\hat{\mathbf{x}}) : \hat{T} \to T_{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4}, \\ \text{with} \qquad & \hat{T} = \left\{0 \le \hat{x}, \hat{y}, \hat{z}, 1 - \hat{x} - \hat{y} - \hat{z} \le 1\right\}, \\ & \lambda_1(\hat{x}, \hat{y}, \hat{z}) = (1 - \hat{x} - \hat{y} - \hat{z}), \quad \lambda_2(\hat{x}, \hat{y}, \hat{z}) = \hat{x}, \\ & \lambda_3(\hat{x}, \hat{y}, \hat{z}) = \hat{y}, \quad \text{and} \quad \lambda_4(\hat{x}, \hat{y}, \hat{z}) = \hat{z}. \end{aligned}$$

We say a tetrahedron T is shape-regular if [6]

$$\frac{h_T}{\rho_T} \le C, \tag{5}$$

where $h_T = \max_{\mathbf{x},\mathbf{y}\in T} |\mathbf{x} - \mathbf{y}|$ is the size of T and ρ_T is the diameter of the maximum ball inscribed in T.

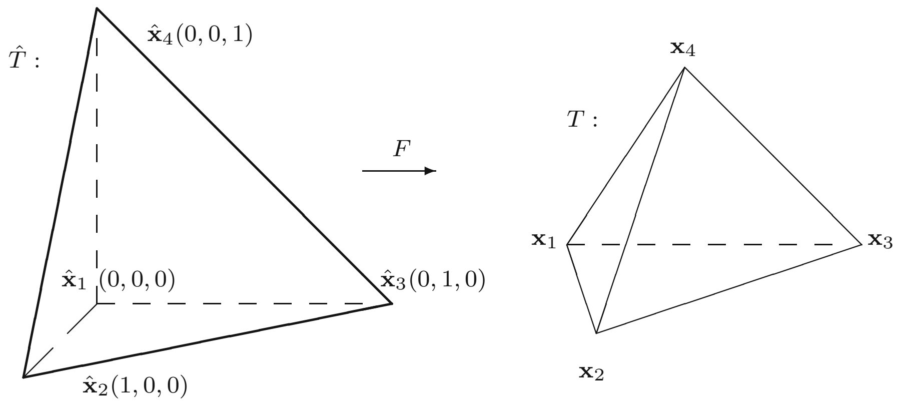

Fig. 2 The reference mapping from tetrahedron $\hat{T}$ to a general tetrahedron T

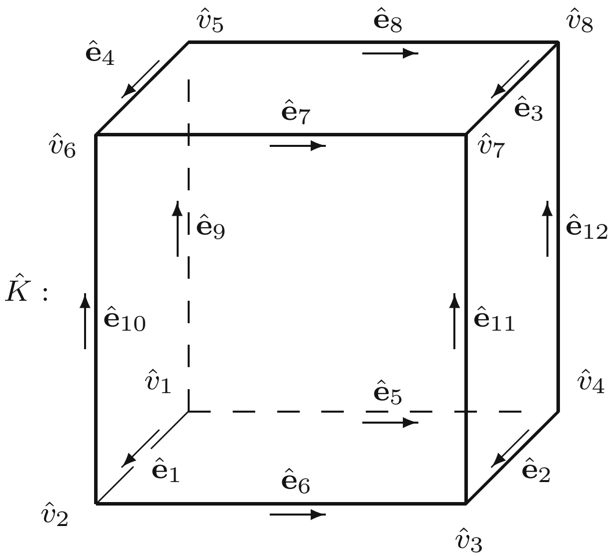

Fig. 3 The 12 edge vectors of a hexahedron

The shape-regularity of a hexahedron is defined by the shape-regularity of the 32 tetrahedra on this hexahedron. For a hexahedron $Q_{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4\mathbf{x}_5\mathbf{x}_6\mathbf{x}_7\mathbf{x}_8}$, we number its 12 edge vectors as in Fig. 3. For each of the 12 edges, we can form two tetrahedra using this edge (as the center edge) and two edges, one at each end of this edge, such that the three edges are not on the same face quadrilateral. We list all 24 such tetrahedra in the first part of (6). The other 8 tetrahedra are the 8 corner tetrahedra, listed in the second part of (7). We say K is shape-regular if all 32 shape-regularity constants in (5) are bounded by a constant.

$$\begin{cases} T_{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_7}, T_{\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4\mathbf{x}_8}, T_{\mathbf{x}_3\mathbf{x}_4\mathbf{x}_1\mathbf{x}_5}, T_{\mathbf{x}_4\mathbf{x}_1\mathbf{x}_2\mathbf{x}_6}, T_{\mathbf{x}_5\mathbf{x}_8\mathbf{x}_7\mathbf{x}_3}, T_{\mathbf{x}_8\mathbf{x}_7\mathbf{x}_6\mathbf{x}_2}, \\ T_{\mathbf{x}_7\mathbf{x}_6\mathbf{x}_5\mathbf{x}_1}, T_{\mathbf{x}_6\mathbf{x}_5\mathbf{x}_8\mathbf{x}_4}, T_{\mathbf{x}_2\mathbf{x}_1\mathbf{x}_5\mathbf{x}_8}, T_{\mathbf{x}_4\mathbf{x}_3\mathbf{x}_7\mathbf{x}_6}, T_{\mathbf{x}_1\mathbf{x}_4\mathbf{x}_8\mathbf{x}_7}, T_{\mathbf{x}_3\mathbf{x}_2\mathbf{x}_6\mathbf{x}_5}, \\ T_{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_5}, T_{\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4\mathbf{x}_6}, T_{\mathbf{x}_3\mathbf{x}_4\mathbf{x}_1\mathbf{x}_7}, T_{\mathbf{x}_4\mathbf{x}_1\mathbf{x}_2\mathbf{x}_8}, T_{\mathbf{x}_5\mathbf{x}_8\mathbf{x}_7\mathbf{x}_1}, T_{\mathbf{x}_8\mathbf{x}_7\mathbf{x}_6\mathbf{x}_4}, \\ T_{\mathbf{x}_7\mathbf{x}_6\mathbf{x}_5\mathbf{x}_3}, T_{\mathbf{x}_6\mathbf{x}_5\mathbf{x}_8\mathbf{x}_2}, T_{\mathbf{x}_5\mathbf{x}_1\mathbf{x}_4\mathbf{x}_6}, T_{\mathbf{x}_3\mathbf{x}_7\mathbf{x}_8\mathbf{x}_2}, T_{\mathbf{x}_4\mathbf{x}_8\mathbf{x}_5\mathbf{x}_3}, T_{\mathbf{x}_2\mathbf{x}_6\mathbf{x}_7\mathbf{x}_1}, \end{cases} \tag{6}$$

$$\begin{cases} T_{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_4\mathbf{x}_5}, \ T_{\mathbf{x}_2\mathbf{x}_3\mathbf{x}_1\mathbf{x}_6}, \ T_{\mathbf{x}_3\mathbf{x}_4\mathbf{x}_2\mathbf{x}_7}, \ T_{\mathbf{x}_4\mathbf{x}_1\mathbf{x}_3\mathbf{x}_8}, \\ T_{\mathbf{x}_5\mathbf{x}_8\mathbf{x}_6\mathbf{x}_1}, \ T_{\mathbf{x}_6\mathbf{x}_5\mathbf{x}_7\mathbf{x}_2}, \ T_{\mathbf{x}_7\mathbf{x}_6\mathbf{x}_8\mathbf{x}_3}, \ T_{\mathbf{x}_8\mathbf{x}_7\mathbf{x}_5\mathbf{x}_4}. \end{cases} \tag{7}$$

Let the polyhedral domain Ω be partitioned by a hexahedral grid $\mathcal{Q}_h = \{K\}$ where each hexahedron K is shape-regular. Each hexahedron K is subdivided into 12 tetrahedra by linking the barycenter $\mathbf{x}_9$ to the 8 vertices and adding a diagonal line on each of 6 face quadrilaterals, cf. the center diagram in Fig. 4. We note that there are two ways to add a diagonal line on a face quadrilateral. Either one is acceptable. But the same selection must be used on the face of the other side hexahedron. This way, we define a shape-regular tetrahedral grid $\mathcal{T}_h$ on Ω. The finite element spaces are defined by

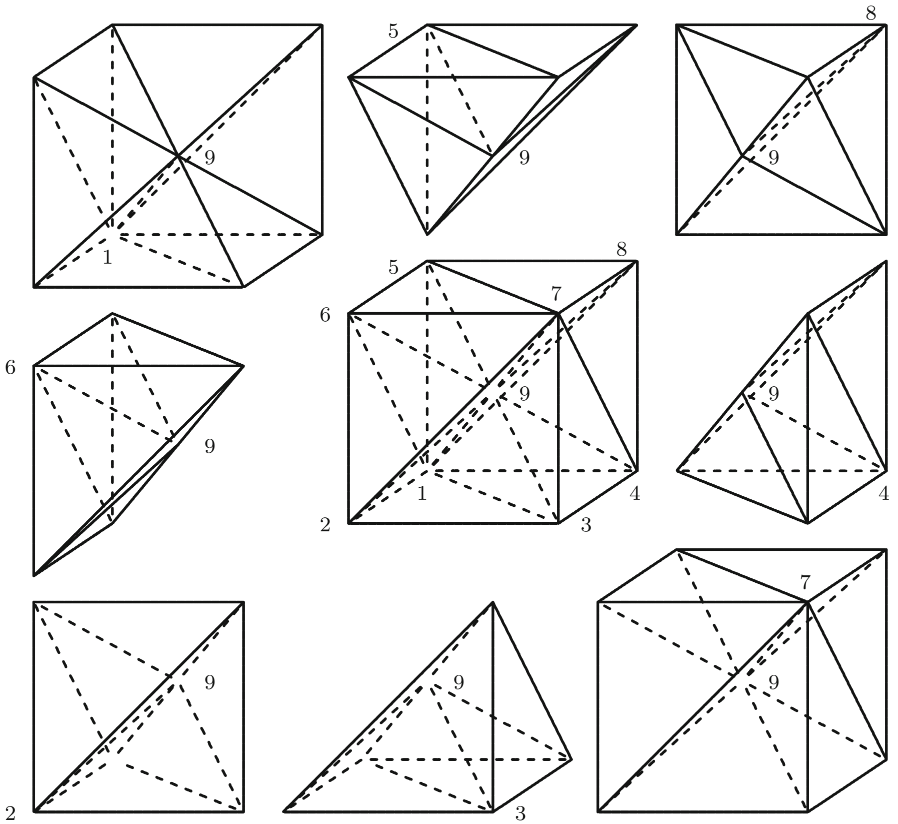

Fig. 4 The twelve-tetrahedra cube, and the tetrahedra around each of eight internal edges

$$\mathbf{V}_h = \left\{ \mathbf{v} \in H_0^1(\Omega)^3 \mid \mathbf{v}|_T \in P_2(T)^3 \quad \text{for all } T \in \mathcal{T}_h \right\}, \tag{8}$$

$$P_h = \left\{ p \in L_0^2(\Omega) \mid p|_T \in P_0(T) \quad \text{for all } T \in \mathcal{T}_h \right\}. \tag{9}$$

The finite element problem is defined as (4).

3 The stability and convergence

Lemma 1 *Let $K \in \mathcal{Q}_h$ which is subdivided into 12 tetrahedra T_i of $\mathcal{T}_h$. For each $p_h \in P_h \cap L_0^2(K)$, there is $\mathbf{u}_h \in \mathbf{V}_{h,0}(K) := \mathbf{V}_h \cap H_0^1(K)^3$ such that*

$$(\operatorname{div} \mathbf{u}_h, p_h) \geq C \|\mathbf{u}_h\|_{H^1} \|p_h\|_0 \tag{10}$$

for some positive constant C independent of K, T, and h.

Proof Given any $p_h \in P_h$, let p_m be the constant value of p_h on the tetrahedron $T_m = T_{ijkl}$. We will show that p_h is a global constant, instead of 12 constants on the 12 tetrahedra, if $(\operatorname{div} \mathbf{u}_h, p_h) = 0$ for all internal $\mathbf{u}_h \in \mathbf{V}_{h,0}(K)$. We use the six mid-edge Lagrange P_2 nodal basis functions only, to test.

We number the 8 vertices of a hexahedron K as shown in Fig. 4. The bary-center vertex is numbered by 9. We further number the 12 tetrahedra as, cf. Fig. 4,

$$T_1 = T_{1489}, T_2 = T_{1589}, T_3 = T_{2379}, T_4 = T_{25y9}, T_5 = T_{1269}, T_6 = T_{1569},$$
$$T_7 = T_{3479}, T_8 = T_{4789}, T_9 = T_{1239}, T_{10} = T_{1349}, T_{11} = T_{5679}, T_{12} = T_{5789}.$$

There are 8 internal edges, linking the center point $\mathbf{x}_9$ to the 8 vertices of the hexahedron. There are two types of edges. E_{19} and E_{79} (denoting the edge with end points 7 and 9) have 6 tetrahedra around each, while the other 6 edges have 4 tetrahedra.

On the patch of 4 tetrahedra around the internal edge E_{59}, cf. Fig. 4, we choose two velocity functions $\mathbf{u}_h$,

$$\mathbf{u}_h = \phi_{59}\mathbf{t}_{57}, \phi_{59}\mathbf{t}_{68}, \tag{11}$$

where ϕ_{59} is the P_2 nodal basis function having value 1 at the mid-edge node of edge E_{59}, and the vector $\mathbf{t}_{57} = \mathbf{x}_7 - \mathbf{x}_6$ is a vector starting at point $\mathbf{x}_6$ and ending at point $\mathbf{x}_7$. Note that each of the 6 patches has exactly one plane face having two triangles. The two edge vectors are diagonal vectors of the plane quadrilateral. To find the nodal basis function, we let λ_{1659} be the volume coordinate which is a linear function, having a value 1 at vertex $\mathbf{x}_9$ and a value 0 on the triangle $\triangle_{165}$. Thus,

$$\phi_{59} = 4\lambda_{1659}\lambda_{1695}, \quad \text{on tetrahedron } T_{1569}.$$

On $T_6 = T_{1569}$, $\operatorname{div} \phi_{59}\mathbf{t}_{57}$ is a linear function, vanishing on the edge E_{16}. Then its integral is the sum of the other two vertex values at $\mathbf{x}_5$ and $\mathbf{x}_9$ times $1/4$ of the volume $|T_{1569}| = |T_6| = V_6$.

$$\begin{aligned}
\int_{T_{1569}} \operatorname{div} \phi_{59}\mathbf{t}_{57} dV &= \left(4\frac{\mathbf{n}_{1569}\cdot\mathbf{t}_{57}}{h_{1569}} + 4\frac{\mathbf{n}_{1965}\cdot\mathbf{t}_{57}}{h_{1965}}\right)\frac{V_6}{4} \\
&= \left(\frac{\mathbf{n}_{1569}\cdot\mathbf{t}_{57}}{h_{1569}} + \frac{\mathbf{n}_{1965}\cdot(\mathbf{t}_{56}+\mathbf{t}_{67})}{h_{1965}}\right)V_6 \\
&= \left(\frac{V_{1567}}{V_{1569}} + \frac{-V_{1569}+V_{1679}}{V_6}\right)V_6 = V_{5679} = V_{11},
\end{aligned}$$

where $\mathbf{n}_{1569}$ is the unit normal vector on triangle $\triangle_{156}$ pointing toward $\mathbf{x}_9$, h_{1569} is the height of a tetrahedron when the base is a triangle with the first three letters, $\triangle_{156}$. We note that if $\mathbf{x}_1\mathbf{x}_9\mathbf{x}_7$ is a straight line, $h_{1967} = 0$ and $V_{1967} = 0$. When it is not a straight line, V_{1967} may have a negative value, but its combination with V_{1567} would be the volume of two real tetrahedra, V_{1569} and V_{5679}.

On $T_{11} = T_{5679}$, the edge $\mathbf{t}_{57}$ is on one outside face triangle, $\triangle_{567}$. Thus,

$$\begin{aligned}\int_{T_{5679}} \operatorname{div} \phi_{59}\mathbf{t}_{57} dV &= \int_{T_{5679}} 4 \operatorname{div} \lambda_{5679}\lambda_{6795}\mathbf{t}_{57} dV\\ &= 4\left(0 + \frac{\mathbf{n}_{6795}\cdot\mathbf{t}_{57}}{h_{6795}}\right)\frac{V_{11}}{4} = -V_{5679} = -V_{11}.\end{aligned}$$

On $T_{5789} = T_{12}$, $\phi_{59} = 4\lambda_{5789}\lambda_{7895}$ and

$$\begin{aligned}\int_{T_{5789}} \operatorname{div} \phi_{59}\mathbf{t}_{57} dV &= \left(4\frac{\mathbf{n}_{5789}\cdot\mathbf{t}_{57}}{h_{5789}} + 4\frac{\mathbf{n}_{7895}\cdot\mathbf{t}_{57}}{h_{7895}}\right)\frac{V_{5789}}{4}\\ &= \left(0 - \frac{V_{5789}}{V_{5789}}\right)V_{12} = -V_{5789} = V_{12}.\end{aligned}$$

On the last tetrahedron $T_{1589=T_2}$, we could compute the integral again with $\phi_{59} = 4\lambda_{1589}\lambda_{1895}$. But, as $\int_{C_{59}} \operatorname{div} \phi_{59}\mathbf{t}_{67} = 0$, where C_{59} is the union of four tetrahedra around the edge E_{59}, the last integral is simply the negative value of the sum of above three integrals.

$$\int_{T_{1589}} \operatorname{div} \phi_{59}\mathbf{t}_{57} dV = -(V_{11} - V_{11} - V_{12}) = V_{12}.$$

With the above four integrals, that $\int_{C_{59}} \operatorname{div} \phi_{59}\mathbf{t}_{57} p_h = 0$ generates a linear equation,

$$p_6 - P_{11} + \frac{V_{12}}{V_{11}}(p_2 - P_{12}) = 0. \tag{12}$$

We repeat the above computation for $\int_{C_{59}} \operatorname{div} \phi_{59}\mathbf{t}_{68}$. Note that $\mathbf{t}_{68}$ and $\mathbf{t}_{57}$ are the two diagonal vectors on the top plane.

$$\begin{aligned}\int_{T_{5679}} \operatorname{div} \phi_{59}\mathbf{t}_{68} dV &= \left(4\frac{\mathbf{n}_{5679}\cdot\mathbf{t}_{68}}{h_{5679}} + 4\frac{\mathbf{n}_{6795}\cdot\mathbf{t}_{68}}{h_{6795}}\right)\frac{V_{5679}}{4}\\ &= \left(0 + \frac{V_{6789}}{V_{6795}}\right)V_{11} = V_{6789}.\\ \int_{T_{1569}} \operatorname{div} \phi_{59}\mathbf{t}_{68} dV &= \left(4\frac{\mathbf{n}_{1569}\cdot\mathbf{t}_{68}}{h_{1569}} + 4\frac{\mathbf{n}_{1695}\cdot\mathbf{t}_{68}}{h_{1695}}\right)\frac{V_{1569}}{4}\\ &= \left(\frac{V_{1568}}{V_{1569}} + \frac{V_{1689}}{V_{1569}}\right)V_6 = V_{C_{59}} - V_{6789} = V_6 + V_2 + V_{5689}.\\ \int_{T_{5789}} \operatorname{div} \phi_{59}\mathbf{t}_{68} dV &= \left(4\frac{\mathbf{n}_{5789}\cdot\mathbf{t}_{68}}{h_{5789}} + 4\frac{\mathbf{n}_{7895}\cdot\mathbf{t}_{68}}{h_{7895}}\right)\frac{V_{5789}}{4}\\ &= \left(0 - \frac{V_{6789}}{V_{5789}}\right)V_{12} = -V_{6789}.\end{aligned}$$

$$\int_{T_{1589}} \operatorname{div} \phi_{59} \mathbf{t}_{68} dV = \left(4 \frac{\mathbf{n}_{1589} \cdot \mathbf{t}_{68}}{h_{1589}} + 4 \frac{\mathbf{n}_{1895} \cdot \mathbf{t}_{68}}{h_{1895}}\right) \frac{V_{1589}}{4}$$
$$= \left(-\frac{V_{1568}}{V_{1589}} - \frac{V_{1689}}{V_{1589}}\right) V_2 = -(V_{C_{59}} - V_{6789}).$$

Combining the four integrals into $\int_{C_{59}} \operatorname{div} \phi_{59} \mathbf{t}_{68} p_h = 0$, we get another linear Eq.

$$p_{11} - P_{12} + \frac{V_6 + V_2 + V_{5689}}{V_{6789}} (p_6 - P_2) = 0. \tag{13}$$

Under our assumption of regular hexahedron, V_{5689} and V_{6789} are positive volumes (of order $O(h^3)$) of the two tetrahedra.

Next, we repeat the above computation for testing functions $\mathbf{u}_h = \phi_{89}\mathbf{t}_{81}, \phi_{89}\mathbf{t}_{54}$ on the patch of four tetrahedra around edge E_{89}:

$$p_{12} - p_2 + \frac{V_1}{V_2}(p_8 - p_1) = 0, \tag{14}$$
$$p_2 - p_1 + \frac{V_6 + V_{12} + V_{1459}}{V_{4589}}(p_{12} - p_8) = 0. \tag{15}$$

On the patch around edge E_{49}, we get two more equations,

$$p_1 - p_8 + \frac{V_7}{V_8}(p_{10} - p_7) = 0, \tag{16}$$
$$p_8 - p_7 + \frac{V_1 + V_{10} + V_{4789}}{V_{3789}}(p_1 - p_{10}) = 0. \tag{17}$$

Working on the next three patches around E_{39}, E_{29} and E_{69}, we get 6 more equations,

$$p_7 - p_{10} + \frac{V_9}{V_{10}}(p_3 - p_9) = 0, \tag{18}$$
$$p_{10} - p_9 + \frac{V_3 + V_7 + V_{1349}}{V_{1249}}(p_7 - p_3) = 0, \tag{19}$$
$$p_9 - p_3 + \frac{V_3}{V_4}(p_5 - p_4) = 0, \tag{20}$$
$$p_3 - p_4 + \frac{V_5 + V_9 + V_{2369}}{V_{3679}}(p_9 - p_5) = 0, \tag{21}$$
$$p_4 - p_5 + \frac{V_6}{V_5}(p_{11} - p_6) = 0, \tag{22}$$
$$p_5 - p_6 + \frac{V_4 + V_{11} + V_{2569}}{V_{1259}}(p_4 - p_{11}) = 0. \tag{23}$$

By the 12 equations, we show p_i are all the same. By Eqs. (12), (14), (16), (18), (20) and (22), it follows that

$$p_6 - p_{11} = \frac{V_{12}}{V_{11}}(p_2 - p_{12}) = \frac{V_{12}}{V_{11}}\frac{V_1}{V_2}(p_8 - p_1) = \frac{V_{12}}{V_{11}}\frac{V_1}{V_2}\frac{V_7}{V_8}(p_{10} - p_7)$$
$$= \cdots = \frac{V_{12}}{V_{11}}\frac{V_1}{V_2}\frac{V_7}{V_8}\frac{V_9}{V_{10}}\frac{V_3}{V_4}\frac{V_6}{V_5}(p_{11} - p_6).$$

Since all these volumes are non-zero, but of order h^3, we conclude

$$p_6 = p_{11}.$$

So all the middle terms in the above equations are zero that

$$p_2 = p_{12}, \quad p_1 = p_8, \quad p_7 = p_{10}, \quad p_9 = p_3, \quad p_4 = p_5.$$

From last Eq. (23), we get now

$$p_4 - p_{11} + \frac{V_4 + V_{11} + V_{2569}}{V_{1259}}(p_4 - p_{11}) = 0.$$

So

$$p_4 = p_{11}.$$

Similarly by (13), (15),(17),(19), and (21),

$$p_6 = p_2, \quad p_{12} = p_8, \quad p_1 = p_{10}, \quad p_3 = p_9, \quad p_9 = p_5.$$

Thus all constants are equal and

$$p_h = p_1 \text{ on } K.$$

Therefore, as we have a finite dimensional system of equations, defined on shape regular tetrahedra of same size h, the above orthogonality guarantees a solution $\mathbf{u}_h$ such that $\int_{T_i} \operatorname{div} \mathbf{u}_h = \int_{T_i} p_h$ and (10) holds.

Theorem 1 *The inf-sup condition* (2) *holds for the finite element spaces* $\mathbf{V}_h$ *and* P_h *defined in* (8) *and* (9)*, respectively. The finite element solution* $(\mathbf{u}_h, p_h)$ *of* (4) *approximates that of* (1) *at the optimal order*

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1} + \|p - p_h\|_{L^2} \le Ch(\|\mathbf{u}\|_{H^2} + \|p\|_{H^1}). \tag{24}$$

Proof The proof of the inf-sup condition is basically by the macro-technique of Stenberg [26]. For a $p_h \in P_h$, cf. [8,11], there is a continuous $\mathbf{u} \in H_0^1(\Omega)^3$ such that

$$\operatorname{div} \mathbf{u} = p_h, \qquad \text{and} \quad \|\mathbf{u}\|_{H^1} \le C\|p_h\|_{L^2}.$$

We define an interpolation of $\mathbf{u}$, $\mathbf{u}_I \in \mathbf{V}_h$. At all vertex nodes $\mathbf{x}_i$ of tetrahedral grid $\mathcal{T}_\langle$, we let $\mathbf{u}_I(\mathbf{x}_i) = I_h u(\mathbf{x}_i)$ where I_h is the Scott-Zhang operator, averaging weak functions [25]. Also, at mid-edge nodes $\mathbf{e}_i$, except those mid-edge nodes also on the digonal-edge of face quadrilateral of hexahedron K of the hexahedral grid $\mathcal{Q}_h$, we let $\mathbf{u}_I(\mathbf{e}_i) = I_h u(\mathbf{e}_i)$. At a mid-edge point $\mathbf{m}_i$, which is also a mid-point on the face-quadrilateral F_j of a hexahedron K, we define

$$\mathbf{u}_I(\mathbf{m}_i) = \frac{1}{\int_{F_j} \phi_i d\mathbf{x}} \int_{F_j} \mathbf{u}(\mathbf{x}) - \sum_{l=1}^{8} \mathbf{u}_I(\mathbf{x}_l)\phi_l(\mathbf{x}) d\mathbf{x},$$

where $\mathbf{x}_l$ are 4 vertexes of F_j and 4 mid-edge points of F_j, and ϕ_l is a Lagrange nodal basis function there. Here we take F_j as the two-piece-flat quadrilateral formed by two triangles of tetrahedra in $\mathcal{T}_h$. That is, $\mathbf{u}_I(\mathbf{m}_i)$ is defined such that

$$\int_{F_j} \mathbf{u} d\mathbf{x} = \int_{F_j} \mathbf{u}_I d\mathbf{x} \quad \text{on all face quadrilaterals } F_j \text{ of } \mathcal{Q}_h. \tag{25}$$

The interpolation is H^1-stable that

$$\|\mathbf{u}_I\|_{H^1} \le C\|\mathbf{u}\|_{H^1}.$$

Let $q_h = p_h - p_{h,1}$ where $p_{h,1} = \sum_{T\in\mathcal{T}_h} \chi_T \int_T \operatorname{div} \mathbf{u}_I/|T|$ and χ_T is the characteristic function supported on T. By (25), on each hexahedron K of $\mathcal{Q}_h$,

$$\begin{aligned}\int_K q_h d\mathbf{x} &= \int_K q_h d\mathbf{x} - \int_K \operatorname{div} \mathbf{u}_I d\mathbf{x} = \int_K q_h d\mathbf{x} - \int_{\partial K} \mathbf{u}_I \cdot \mathbf{n} dS \\ &= \int_K q_h d\mathbf{x} - \int_{\partial K} \mathbf{u} \cdot \mathbf{n} dS = \int_K q_h - \operatorname{div} \mathbf{u} d\mathbf{x} = 0.\end{aligned}$$

So $q_h \in P_h \cap L_0^2(K)$ on each hexahedron $K \in \mathcal{Q}_h$. By Lemma 1, there is an $\mathbf{u}_{h,0} \in \mathbf{V}_h$, and $\mathbf{u}_{h,0} \in \mathbf{V}_h \in V_{h,0}(K)$ for all $K \in \mathcal{Q}_h$, such that

$$\int_T \operatorname{div} \mathbf{u}_{h,0} = \int_T q_h, \quad \text{and} \quad \|\mathbf{u}_{h,0}\|_{H^1} \le C\|q_h\|_{L^2}.$$

Together, letting $\mathbf{u}_h = \mathbf{u}_I + \mathbf{u}_{h,0}$, we have

$$\begin{aligned}(\operatorname{div} \mathbf{u}_h, p_h) &= (\operatorname{div} \mathbf{u}_I + \operatorname{div} \mathbf{u}_{h,0}, p_{h,1} + q_h) \\ &= (\operatorname{div} \mathbf{u}_I, p_{h,1}) + (\operatorname{div} \mathbf{u}_{h,0}, q_h) = \|p_{h,1}\|_{L^2}^2 + \|q_h\|_{L^2}^2 \ge \frac{1}{2}\|p_h\|_{L^2}^2\end{aligned} \tag{26}$$

and

$$\|\mathbf{u}_h\|_{H^1} \le C(\|\mathbf{u}_I\|_{H^1} + \|\mathbf{u}_{h,0}\|_{H^1}) \le C(\|\mathbf{u}\|_{H^1} + \|q_h\|_{L^2}) \le C\|p_h\|_{L^2}. \tag{27}$$

Combining (26) and (27), shows that the inf-sup condition (2) is obtained. Thus, by the standard theory [11],

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{H^1} + \|p-p_h\|_{L^2} &\le C\left(\inf_{\mathbf{v}_h\in\mathbf{V}_h}\|\mathbf{u}-\mathbf{v}_h\|_{H^1} + \inf_{q_h\in P_h}\|p-q_h\|_{L^2}\right)\\ &\le Ch\left(\|\mathbf{u}\|_{H^2} + \|p\|_{H^1}\right).\end{aligned}$$

The proof is completed.

4 Numerical tests

We solve the Stokes problem (1) on the unit cube $\Omega = (0,1)^3$. The right-hand side function $\mathbf{f}$ in (1) is chosen such that the exact solution is

$$\begin{cases} \mathbf{u} = 2^{12}\nabla\times\begin{pmatrix}0\\ (x-x^2)^2(y-y^2)^2(z-z^2)^2\\ (x-x^2)^2(y-y^2)^2(z-z^2)^2\end{pmatrix},\\ p = \dfrac{2^{14}}{9}(x-3x^2+2x^3)(y-3y^2+2y^3)(z-z^2)^2. \end{cases} \tag{28}$$

On the first level, the hexahedral grid consists of only one cube, the domain Ω. Then the tetrahedral grid is made up by 12 tetrahedra, shown in Fig. 1. We apply the multigrid refinement [31] to obtain higher level grids. We plot the first three grids in Fig. 5.

The computational results are listed in Table 1. We can only prove the first order of convergence. But it is likely there is a super-convergence for the P_2 velocity finite element. So we obtain a second order of convergence for the velocity in the H^1 norm and a third order of convergence in the L^2 norm. This is not possible on general/non-uniform grids because the error is limited by the 0th order approximation of pressure. Surprisingly we have a convergence of third order for the P_0 pressure finite element

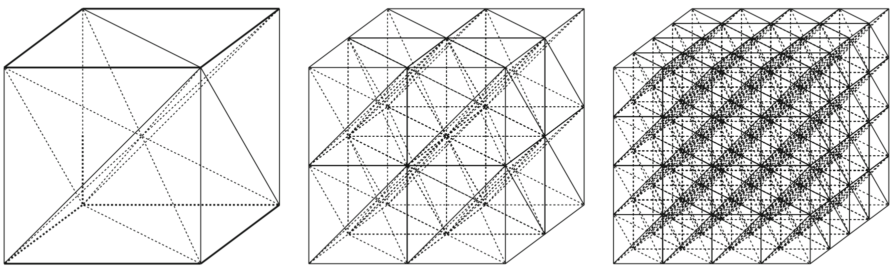

Fig. 5 The first three levels of grids for the solution (28) in Table 1

Table 1 The errors, where $\mathbf{e_u} = \mathbf{u}_I - \mathbf{u}_h$, and the orders of convergence, by the $C_0 P_2 - P_0$ element (8)–(9) on the sub-hexahedron tetrahedron grids

	$\|\mathbf{e_u}\|_{L^2}$	h^n	$\|\mathbf{e_u}\|_{H^1}$	h^n	$\|p_I - p_h\|_{L^2}$	h^n	dim V_h	dim P_h
1	0.7949	0.0	7.3961	0.0	298.7512	0.0	105	12
2	0.3183	1.3	3.8922	0.9	15.7439	4.2	567	96
3	0.0369	3.1	1.1258	1.8	3.8006	2.1	3723	768
4	0.0045	3.0	0.3017	1.9	0.3565	3.4	27027	6144
5	0.0006	2.9	0.0798	1.9	0.0259	3.8	206115	49152

solution, from the numerical data. We believe the order of convergence would be reduced to 2 eventually, due to initial large error of pressure, a common phenomenon for discontinuous pressure approximation. But even with order two, the P_0 element is one order super-convergent. Though the $P_2 - P_0$ combination is not optimal, the numerical results show the method is of optimal order, on uniform grids.

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